SALLY TRUEMAN

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HIGH SCHOOL ALGEBRA

MACMILLAN'S CANADIAN SCHOOL SERIES

THE

NEW BRUNSWICK SCHOOL ALGEBRA

BY

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PREFACE

THIS text covers the work prescribed for entrance to the Universities

The book is written from the standpoint of the pupil, and in such a form that he will be able to understand it with a minimum of assistance from the teacher. The question method is frequently used in developing the theory. The purpose of this is to lead the pupil to think for himself.

The close connection between algebra and arithmetic is constantly kept in view, and in many cases the arithmetical and algebraic processes are shown in parallel columns.

There are numerous diagrams for the purpose of illustrating the theory, and algebraic methods are applied to many of the theorems which the pupil meets in elementary geometry.

Special emphasis is placed upon the verification of results. In the past, sufficient attention has not been given to this important part of mathematical work.

Provision is made for oral work, many of the exercises being introduced by a number of oral examples for use in class.

The equation and the solution of simple problems are introduced in the second Chapter. It is hoped that the pupil will thus become interested much earlier in the work.

Long multiplications and divisions are not included in the work of the first year. They are difficult for the beginner and of little interest, as there is not much to offer in the way of practical illustrations.

Chapter X., with which the pupil would begin the second year's work, contains a thorough review of the simple rules. Here the more complicated processes are dealt with.

The graphical work is introduced naturally in illustrating the negative quantity and in the solution of equations. Only graphs which can be drawn with the ruler and compasses are included in the book.

More attention is given to methods of inspection in the extraction of roots. The long process for cube root is eliminated, as cube root is not now required in arithmetic.

The division method of finding highest common factor has been discarded, as it is usually performed mechanically and not understood by pupils. The elimination method which is used will be found easy to apply with expressions which are not too complicated. Finding the highest common factor of expressions of the fourth or higher degrees is of little algebraic value, and few examples of such problems will be found in the book.

The review exercises at the end of each Chapter will be found useful, particularly for the purpose of reviewing the work of a previous term.

On the recommendation of experienced teachers the answers are not given to simple examples, or to such examples as the pupil can verify without difficulty.

CONTENTS

CHAPTER I

Algebraic Notation . . . . PAGE 1

Arithmetical and Algebraic Signs and Symbols (1). Fundamental Laws (3). Factor and Product (6). Power and Index (7). Terms (8). Coefficient (9). Addition and Subtraction of like Terms (9). Use of Brackets (11). Review Exercise (13).

CHAPTER II

Simple Equations 16

Meaning of Equation, Solving an Equation, Root of an Equation (16). Axioms used (18). Verifying Results (19). Problems solved by Equations (21). Review Exercise (25).

CHAPTER III

Positive and Negative Numbers 28

Graphical Representation of Positive and Negative Numbers (28). Concrete Examples of Negative Numbers (30). Signs of Operation and Signs of Quality (32). Absolute Value (32). Review Exercise (34).

CHAPTER IV

Addition and Subtraction 36

Addition of Quantities with like Signs (36). Compound Addition (37). Addition with unlike Signs (39). Indicated Additions (41). Subtraction the Inverse of Addition (42). Rule for Subtraction (43). Removal of Brackets (45). Review Exercise (47).

CHAPTER V

PAGE

Multiplication and Division 49

Multiplication of Simple Positive Quantities (49). Index Law for Multiplication (49). Rule of Signs (50). Compound Multiplication (53). Verifications (55). Division by a Simple Quantity (57). Index Law for Division (58). Review Exercise (60).

CHAPTER VI

Simple Equations (*continued*) 62

Equation and Identity (62). Transposing Terms (63). Simple Fractional Equations (67). Steps in the Solution of an Equation (67). Problems (69). Algebraic statements of Arithmetical Theorems (74). Review Exercise (75).

CHAPTER VII

Simultaneous Equations 78

Equations with two Unknowns (78). Method of Solution (79). Elimination (80). Fractional Simultaneous Equations (82). Problems (83). Review Exercise (85).

CHAPTER VIII

Type Products and Simple Factoring 88

Monomial Factors (88). Product of two Binomials (89). Factors of Trinomials (90). Radical Sign (92). Square of a Binomial (94). Square Root of a Trinomial (95). Difference of two Squares (97). Numerical Applications (99). Review Exercise (101).

CHAPTER IX

PAGE

Simple Applications of Factoring 102

Highest Common Factor (103). Algebraic Fractions (104). Lowest Terms (105). Multiplication and Division of Fractions (106). Lowest Common Multiple (107). Addition and Subtraction of Fractions (108). Mixed Expressions (110). Review Exercise (111).

CHAPTER X

Review of the Simple Rules 114

Brackets (114). Collecting Coefficients (117). Multiplication with Detached Coefficients (118). Division by a Compound Quantity (121). Verifying Division (122). Inexact Division (124). Review Exercise (125).

CHAPTER XI

Factoring *(continued)* 128

Factors by Grouping (128). Complete Squares (130). Difference of Squares (132). Incomplete Squares (135). Trinomials (137). Sum and Difference of Cubes (140). The Factor Theorem (141). Easy Quadratic Equations (144). Review Exercise (146).

CHAPTER XII

Simultaneous Equations *(continued)* 149

Elimination by Substitution and by Comparison (149). Equations with three Unknowns (152). Special Forms of Equations (154). Solution of Problems (156). Review Exercise (161).

CHAPTER XIII

PAGE

Geometrical Representation of Number 165

Function of x (165). Variables and Constants (165). Arithmetical Graphs (166). The Axes (169). Graph of an Equation (170). Coordinates (170). Plotting Points (171). Linear Equation (173). Graphical Solution of Simultaneous Equations (175). Special Forms of Equations (176). Review Exercise (177).

CHAPTER XIV

Highest Common Factor and Lowest Common Multiple . 180

By Factoring (180). By Elimination (183). Product of the H.C.F. and L.C.M. (185). Review Exercise (187).

CHAPTER XV

Fractions 188

Changes in the Form of a Fraction (188). Lowest Terms (190). Addition and Subtraction (191). Special Types (193). Cyclic Order (194). Multiplication and Division (197). Complex Fractions (199). Review Exercise (201).

CHAPTER XVI

Fractional Equations 204

Cross Multiplication (204). Special Forms of Fractional Equations (206). Literal Equations with one Unknown (209), with two Unknowns (212). Review Exercise (214).

CHAPTER XVII

PAGE

Extraction of Roots 216

Square Root by Inspection (216), by the Formal Method (217). Verifying Results (218). Cube of a Binomial (222). Cube Root (223). Higher Roots (224). Review Exercise (226).

CHAPTER XVIII

Quadratic Surds 228

Surd and Rational Quantities (228). Mixed and Entire Surds (229). Like Surds (231). Addition and Subtraction (231). Square Root Table (232). Conjugate Surds (233). Rationalizing a Denominator (235). Surd Equations (237). Review Exercise (238).

CHAPTER XIX

Quadratic Equations 240

Standard Form (241). Solution by Factoring (242), by Completing the Square (244). Irrational Roots (247). Inadmissible Solutions (248). Review Exercise (250).

CHAPTER XX

Ratio and Proportion 253

Methods of Comparing Magnitudes (253). Comparison of Ratios (254). Proportion (256). Finding a Ratio by Solving an Equation (257). Mean Proportional (261). Ratio Theorems (263). Review Exercise (266).

CHAPTER XXI

PAGE
The General Quadratic Equation 268
Solution of Literal Quadratics (268). Solution by Formula (270). Imaginary Roots (271). Equations Solved like Quadratics (273). Review Exercise (276).

CHAPTER XXII

Simultaneous Quadratics 279
Three Types of Simultaneous Quadratics (279). Special Methods (284). Graphical Solutions (288). Review Exercise (290).

CHAPTER XXIII

Indices 293
The Index Laws (294). Fractional, Zero and Negative Indices (297). Operations with Fractional and Negative Indices (301). Contracted Methods (303). Review Exercise (305).

CHAPTER XXIV

Surds and Surd Equations 308
Surds of different Orders (308). Varying Forms of Surds (309). Surd Equations (312). Extraneous Roots (313). Square Root of a Binomial Surd (317). Imaginary Surds (320). Impossible Problems (323). Review Exercise (324).

CHAPTER XXV

Theory of Quadratic Equations 327
Sum and Product of the Roots (327). Reciprocal Roots (328). Functions of the Roots (331). Character of the Roots (335). The Discriminant (337). Factors of the Quadratic Expression (338). Review Exercise (341).

CHAPTER XXVI

PAGE

Miscellaneous Theorems and Examples . . . 344

Factors of the Product of two Trinomials (344). Sum and Difference of Cubes (346). Factors by Grouping (349). The Factor Theorem (351). Symmetry (354). Factoring by Symmetry (355). Identities (359). Review Exercise (362).

CHAPTER XXVII

Progressions 365

Series (365). Arithmetic Progression (365). General Term (366). Sum (368). Harmonic Progression (373). General Term (373). Geometric Progression (376). General Term (376). Sum (380). Limit of the Sum to Infinity (381). Review Exercise (383).

CHAPTER XXVIII

Permutations and Combinations 385

Value of ${}_nP_r$ (387). Permutations with Repetitions (390). Value of ${}_nC_r$ (392). Greatest Value of ${}_nC_r$ (395). Review Exercise (396).

Answers 401

HIGH SCHOOL ALGEBRA

HIGH SCHOOL ALGEBRA

CHAPTER I

ALGEBRAIC NOTATION

1. **Use of Arithmetical Signs.** In arithmetic, signs are used to abbreviate the work. In algebra the same signs are used, with the same meanings and for the same purpose.

EXERCISE 1

Write the following statements in the shortest way you can, using the signs and symbols with which you are familiar in arithmetic.

1. Two and two make four.

2. The sum [illegible]e, ten and twenty is thirty-five.

3. Six and [illegible] the same as four and six.

4. Seven times eight is the same as eight times seven.

5. The difference between twelve and five is seven.

6. Ten exceeds six by four.

7. The excess of twenty over fifteen is five.

8. The defect of thirty from a hundred is seventy.

9. Thirty-six divided by four is nine.

10. Three score and ten is seventy.

11. One half of the sum of seven and five is six.

12. The sum or the product of three, five and seven is the same in whatever order they are written.

13. Three multiplied by four is twelve, therefore twelve divided by three is four.

14. The square of four is sixteen, therefore the square root of sixteen is four.

2. Algebraic Symbols. In the preceding exercise you have used symbols to represent the numbers stated and signs to show the operations performed on those numbers.

In algebra, symbols are used more extensively than in arithmetic.

If the length of this line be measured it will be found to be two inches. But without measuring it, we may say that the measure of its length is some definite number which might be represented by the letter a.

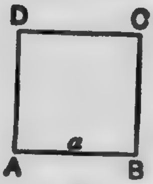
The measure of the length of another line might be represented by b. The cost of an article might be c cents, or the cost of a farm might be x dollars, or the weight of a stone might be m pounds.

Here a, b, c, x, m are **algebraic number-symbols**, or briefly algebraic numbers.

The symbols 1, 2, 3, etc., used to represent numbers in arithmetic are called **arithmetical number-symbols** or arithmetical numbers.

In algebra the number symbols of arithmetic are also used. For the present, when letters are used to represent numbers, it will be understood that each letter represents some integral or fractional number.

3. Signs of Multiplication. In this square the measure of the length of the side AB is a. What is the measure of the length of BC; of CD; of $AB+BC$; of $AB+BC+CD$?



The measure of the perimeter (sum of all the sides) is $a+a+a+a$ or 4 times a or $4\times a$.

In algebra, $4\times a$ or $a\times 4$ is usually written $4a$, the sign of multiplication being understood. It is also written $4\,.\,a$, the dot representing multiplication.

Thus, $4\times a=4\,.\,a=4a$, and as in arithmetic, is a short way of writing $a+a+a+a$.

Thus, if $a=6$, the measure of the perimeter of the square is $6+6+6+6=4\times 6=24$.

It will be observed that in algebra the multiplication of a and 4 is only indicated in the form $4a$, while in arithmetic it may be actually performed as in the result 24.

The pupil must recognize the difference between 24 (*twenty-four*) and the product of 2 and 4 or 2×4 or $2 \,.\, 4$. When two numerical quantities are to be multiplied, the sign of multiplication must be used, so that as stated, 24 may be distinguished from 2×4. When both factors are not numerical as $4 \times a$ or $a \times b$, the sign is omitted and these are written in the form $4a$, ab.

4. Signs of Division. As in arithmetic, the quotient obtained by dividing one number by another may be written in the fractional form.

In arithmetic the division may be actually performed, as in $6 \div 3$, which may be written $\frac{6}{3}$ or 2, but it is frequently only indicated as in $6 \div 7$, which is written $\frac{6}{7}$.

So in algebra, the quotient obtained on dividing a by b, or $a \div b$, is written $\frac{a}{b}$, and here, as in multiplication, the division can only be indicated unless the numerical values of a and b are known.

5. Some Fundamental Laws. Since the letters used in algebra represent arithmetical numbers, all the laws of arithmetic must be true also in algebra.

	In arithmetic.		*In algebra.*
(1)	$7+3=3+7.$ $6+2+5=6+5+2=2+5+6.$	(1)	$a+b=b+a.$ $a+b+c=a+c+b=b+c+a.$
(2)	$3\times 5=5\times 3.$ $2\times 4\times 3=2\times 3\times 4=3\times 4\times 2.$	(2)	$ab=ba.$ $abc=acb=cba.$
(3)	$10+5-2=10-2+5.$ $10-5-2=10-2-5.$	(3)	$a+b-c=a-c+b.$ $a-b-c=a-c-b.$
(4)	$3\times 10\div 5=3\div 5\times 10=10\div 5\times 3$	(4)	$a\times b\div c=a\div c\times b=b\div c\times a.$

From (1) and (2) it follows that *the sum or the product of several numbers is independent of the order in which they are written.*

From (3) and (4) it follows that *a series of additions and subtractions, or of multiplications and divisions, may be made in any order.*

In finding the numerical value of an expression like $3a+4b-2c$ for given values of a, b and c, the operations are performed in the same order as in arithmetic, the multiplications being performed first and then the additions and subtractions in any order.

Thus, when $a=2$, $b=3$, $c=1$,

$$3a+4b-2c=3\times2+4\times3-2\times1=6+12-2=16.$$

Similarly, for the same values of a, b, c,

$$\frac{ab+bc}{a+b}=\frac{2\times3+3\times1}{2+3}=\frac{6+3}{5}=\frac{9}{5}.$$

NOTE.—Many of the examples in the following exercise may be taken orally. The pupil, however, is advised to write the algebraic forms so that he may thereby become familiar with them.

EXERCISE 2

1. When $a=6$, what are the numerical values of:

$$3a,\ \frac{1}{2}a,\ \frac{a}{3},\ \frac{5}{6}a,\ \frac{12}{a},\ \frac{2}{a},\ \frac{5a}{3}?$$

2. When $x=5$ and $y=3$, what are the values of:

$$x+y,\ x-y,\ xy,\ 3x+2y,\ 2x-3y,\ \tfrac{1}{3}xy?$$

3. When $m=4$, $n=6$, $r=2$, find the values of:

$$m+n+r,\ m+r-n,\ mn+mr,\ mr-n,\ 4n-3m-6r.$$

4. Express algebraically the sum, the difference and the product of a and b. What are their values when $a=8$ and $b=3$?

5. The quantities a, b and c are to be added together. Express the sum algebraically. What is its value when $a=6$, $b=4$, $c=12$?

6. When a is divided by b the quotient is expressed in the form $\frac{a}{b}$. When c is added to the quotient of x by y, how is the result expressed? What is its value when $x=12$, $y=4$, $c=10$?

7. A boy has p marbles; he wins q marbles and then loses r marbles. How many has he now? How many if $p=5$, $q=11$, $r=4$?

8. When $a=4$ and $b=5$, find the numerical value of

$$12a-5b+6a-7b+10.$$

9. The sides of a triangle are a, b and c; express algebraically the perimeter and the semi-perimeter. What do they become if $a=13$, $b=14$, $c=15$?

10. Find the cost of 8 articles at 5 cents each; of 7 articles at x cents each; of x yards of cloth at b cents a yard; of m tons of coal at n dollars a ton.

11. How many cents are there in 4 dollars; in x dollars; in x dollars and y cents; in a quarters and b ten-cent pieces ?

12. Find the number of inches in 2 yards; in 3 feet and 7 inches; in a yards; in b feet; in x feet and y inches; in m yards n feet and p inches.

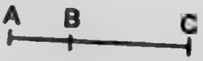
13. What operations are to be performed to find the numerical value of $ma+nb$, when $a=2$, $b=5$, $m=3$, $n=6$? What is the value ?

14. What operations are to be performed to find the value of $\frac{x+y}{a+b}$ when $a=5$, $b=6$, $x=15$, $=7$? What is the value ?

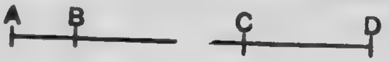
15. By varying the order of the letters, in how many ways can you write $a+b+c$?

16. In how many different ways can you write xyz ?

17. In the figure, BC is twice as long as AB. If AB is l units in length, what is the length of BC ? of AC ?



18. In the figure, BC is three times as long as AB and CD is twice as long as AB. If AB is x units in length, what are the lengths of BC ? CD ? BD ? AD ?



19. In the following statements c represents the cost of an article, s the selling price, and g the gain:

$$(1)\ s-c=g,\ (2)\ c+g=s,\ (3)\ s-g=c.$$

Read them and explain their meanings.

20. What is the next integer above 27 ? The next below 27 ? What is the next integer above n ? The next below n ?

21. If n is an even integer, what is the next even integer above it and the next even integer below it ?

22. If x is any number, what is the number which is 5 greater than x? 5 less than x?

23. A boy is 10 years old. How old will he be in 6 years? In m years? How old was he 4 years ago? n years ago?

24. A man is x years old. How old will he be in n years? How old was he m years ago? In how many years will he be three times as old as he is now?

25. A boy was p years old 3 years ago. How old will he be 15 years from now?

26. Explain the difference between $\frac{a+b}{c}$, $a+\frac{b}{c}$ and $a\,.\,\frac{b}{c}$. What are their values when $a=6$, $b=9$, $c=3$?

27. The side of one square is a and of a smaller one is b. Indicate the difference in their perimeters. What is the difference if $a=10$ and $b=6$?

28. The sides of one rectangle are a and b, and of another are c and d. Indicate the difference in their areas, (1) when the first rectangle is the larger, (2) when the second is the larger.

29. What arithmetical number does $10x+y$ represent when $x=5$, $y=3$? When $x=7$, $y=9$?

30. When $a=3$, $b=4$, $c=5$, $d=0$, find the values of:

(1) $10a+4b-5c+3d$. (2) $5ab+2cd-3ac$.

(3) $\frac{1}{3}ac+\frac{1}{4}bc-\frac{2}{3}ad$. (4) $\frac{6a-2b+3c-d}{2a+b-c+d}$.

6. Factor and Product. When numbers are multiplied together the result is called the product, and the numbers which were multiplied are called the factors of the product.

Thus, $3\times5=15$, therefore the factors of 15 are 3 and 5, so $a\times b=ab$, therefore the factors of ab are a and b.

The factors of $3x$ are 3 and x. The 3 is called a **numerical** factor and the x, a **literal** factor.

Just as 12 may have different sets of factors as 3×4, 2×6, $2\times2\times3$, so $3xy$ has the factors $3\times xy$, $3x\times y$, $x\times3y$ or $3\times x\times y$.

The **prime** factors of 12 are 2, 2 and 3, and the **simplest** factors of $3xy$ are 3, x and y.

An index or exponent is a small number written to the right and above another number to indicate how many times that number is taken as a factor.

In whatever order the factors are written the product is the same, but it is usual to write the numerical factor first and the literal factors in alphabetical order.

7. **Power and Index.** What is the area of a square whose side is 7 inches in length ? The measure of the area of the square in art. 3 is $a \times a$, which is written a^2, and is read "a square," or "a to the second power."

The product when 2 a's are multiplied together is called the **power**, and the 2 is called the **index** or **exponent** of the power.

If the edge of a cube is 6 inches, what is the sum of all the edges ? What is the area of each face of the cube ? What is the area of all the faces ? What is the volume of the cube ?

If the edge of a cube is a, the sum of all the edges is $12a$. The area of each face is a^2, and of all the faces is $6a^2$.

The volume is $a \times a \times a$ or a^3, which is read "a cube," or "a to the third power."

The pupil must distinguish between $3a$ and a^3. The former means $3 \times a$, and the latter $a \times a \times a$.

Thus, if $a=5$, $\quad 3a=3\times 5=15$,
but $\quad a^3=5\times 5\times 5=125$.

EXERCISE 3 (1-14, Oral)

1. What are the prime factors of 35, of 42, of 75 ?

2. What are the simplest factors of $5xy$, of $6mn$?

3. Express $3abc$ as the product of two factors in four different ways.

4. Give two common factors of $15ab$ and $25bc$.

5. Find the values of $3^2\times 2^2$, $10^2\times 5^2$, $2^4\times 5$, $3^2\times 2^2\times 5$.

6. Using an index, express 100 as a power of 10, 16 as a power of 2, 27 as a power of 3, 625 as a power of 5.

7. What is a short way of writing

$$a+a\,? \quad a+a+a\,? \quad a\times a\,? \quad a\times a\times a\,? \quad aaaa\,?$$

8. What is the area of a square whose side is 6 inches ? whose side is x inches ?

Base is the number which is Taken as a factor.

Power is the value of an expression.

9. What is the volume of a cube whose edge is 3 inches? whose edge is m inches?

10. When $a=4$, what is the value of a^2? of $2a$? What is their difference?

11. When $x=2$, what is the difference between x^3 and $3x$?

12. What is the difference between "x square" and "twice x" when $x=11$?

13. If $m=10$, what is the difference between the square of $3m$ and three times the square of m?

14. The side of a square is x inches and of a smaller one is y inches. What is the sum of their areas? What is the difference? What do these results become when $x=10$ and $y=6$?

15.* If $x=6$ and $y=2$, find the numerical values of

$$3x^2,\ x^2+y,\ x+y^2,\ x^2-y^2,\ 2x^2-3y^2.$$

16. Find the values of x^3+x^2+x for the following values of x: $x=1, 2, 3, 0$.

17. If $y=4x^2-7$, find the value of y if $x=2$, if $x=3$, if $x=2\frac{1}{2}$.

18. The unit of work is "a day's work," that is, the work which one man can do in one day. How many units of work can 3 men do in 5 days? 6 men in x days? m men in n days? a men in a days?

19. If $a=3$, $b=2$, $c=1$, find the quotient when $a^2+b^2+c^2$ is divided by $2a+b-c$.

20. Show that x^3+26x has the same value as $9x^2+24$ when $x=2$ or 3 or 4.

21. If $x=10$ and $y=5$, how much greater is x^2+y^2 than $2xy$?

22. If d represents the diameter of a circle and c the circumference, we know that $c=3\frac{1}{7}d$. Find c when $d=14$. Find d when $c=22$.

23. If A represents the area of a circle and r the radius, $A=3\frac{1}{7}r^2$. Find A when $r=7$; when $r=14$.

24. By arranging the factors in the most suitable order, find the values of $2^4 . 5^3$, $25^2 . 4^3$, $125 . 2^5$.

8. **Terms of an Expression.** The parts of an algebraic expression which are connected by the signs of addition or subtraction are called the terms of the expression.

A coefficient [illegible]

Thus, the expression $2a+3b$ has two terms, and the expression $4x^3-3xy-y^2$ has three terms.

Quantities which are connected by the signs of multiplication or division are not different terms.

Thus, $4ax$ is only one term, so is $\frac{a^2}{b}$.

9. Coefficient. In the product $4x$, 4 is called the **coefficient**, or co-factor, of x. In ab, a is the coefficient of b and b is the coefficient of a.

The 4 is a **numerical** coefficient, and the a or b is a **literal** coefficient.

In any product, any factor is called the coefficient of the rest of the product.

Thus, in $5abx$, 5 is the coefficient of abx, $5a$ is the coefficient of bx, and $5ax$ is the coefficient of b.

In any term where the numerical coefficient is not stated, the coefficient 1 is understood.

Thus, in xy the numerical coefficient is 1.

10. Addition and Subtraction of Like Terms. When terms do not differ or differ only in their numerical coefficients, they are called **like terms**.

Thus, $2ab$, $5ab$, $\frac{1}{2}ab$ are like terms, but $3a$, $4b$, $6ab$ are unlike terms.

In arithmetic, quantities which have the same denominations may be added or subtracted.

Thus,

$$3\text{ft.}+4\text{ft.}-2\text{ft.}=5\text{ft.}$$
$$\$12-\$10+\$8-\$3=\$7.$$

We cannot add or subtract quantities of different denominations, unless we can first reduce them to the same denomination.

Similarly, in algebra, like terms may be added or subtracted.

Thus,

$$5a+2a=7a,\ 5a-2a=3a,$$
$$6ab+5ab-3ab=11ab-3ab=8ab,$$
$$8x^2-6x^2+9x^2-2x^2=17x^2-8x^2=9x^2.$$

In the last example we may, of course, perform the operations in the order in which they occur and obtain the same result.

Unlike terms can not be added or subtracted.

Thus, the sum of $3a$ and $5b$ can be indicated in the form $3a+5b$, but they can not be combined into a single term unless the numerical values of a and b are given.

EXERCISE 4 (1-8, Oral)

1. What is the numerical coefficient of each term in the expression
$$5a^3+a^2+\frac{1}{2}a\ ?$$

2. What is the sum of the numerical coefficients in
$$2x^2+3xy+x+y\ ?$$

3. Which are like terms in the expression
$$5a^2+2b-3a+7b-4a^2\ ?$$

4. In $6bcy$, what is the coefficient of bcy ? of cy ? of by ? of b ?

5. What is the sum of :

(1) $2a, 3a, 4a$. (2) $5m, \frac{1}{4}m, \frac{3}{4}m$.
(3) $4a^2, 7a^2, 5a^2$. (4) $3xy, 4xy, 2xy$.

6. If $x=2$, find the numerical value of the sum of $3x^2$ and $4x^2$ in two different ways and compare the results.

7. Simplify the expression $3a+8b+2a+b+a+3b$ by combining like terms.

8. Express in as simple a form as possible :

(1) $5m+7m-3m-2m$. (2) $6ab-3ab+2ab-ab$.
(3) $3x+a+2x+a$. (4) $15a+10b-7a+4b$.

9.* Combine the like terms in the expression :
$$2x+7y+5z-x+2y-3z+3x-4y-z$$
and find its value when $x=3$, $y=5$, $z=10$.

10. If $a=6$, find the value of
$$15a^2-10a^2-3a^2+8a-5a-20.$$

11. What arithmetical number does $100a+10b+c$ represent when $a=2$, $b=3$, $c=4$? When $a=9$, $b=5$, $c=7$?

12. Simplify $2x^2+3x+7-x^2+11x-2-x^2-4x+5$.

13. A man walks $4x$ feet East, then x feet West, then $3x$ feet East, then $5x$ feet West. How far is he now from the starting point and in what direction from it ?

Rule: Do brackets first, multiplication and division in the order they occur, addition and subtraction in the order they occur.

14. A man began to work for a firm on a salary of x dollars a year. If his salary for each year was double the salary for the preceding year, how much did he earn in four years ?

15. If $x+3x+5x$ is equal to 72, what is the value of x ? How do you know that your answer is correct ?

16. Write in the shortest form you can

$$aaa+aaa+aa+aa+a+a+a.$$

17. Find the average of (1) 10, 8, 15, (2) $3x$, $7x$, $5x$.

11. **Use of Brackets.** In algebra brackets are used for the same purpose and with the same meanings as in arithmetic.

In finding the value of $10+8+5$, we may perform the additions in any order, but if we write it $10+(8+5)$, it is understood that the 8 and 5 are first to be added and the sum of 10 and the result is to be taken.

Similarly, $a+(b+c)$ means that the sum of the numbers represented by b and c is to be added to the number represented by a.

In the expression $7+5\times2$, the multiplication is to be performed first, and then the addition. If, however, we wish the value of $(7+5)\times2$, we must add the 7 and 5 before multiplying by 2.

Although $10+(8+5)$ is equal to $10+8+5$, it is clear that $(7+5)\times2$ is not equal to $7+5\times2$, the former being equal to 24 and the latter 17.

When a is to be multiplied by b, the sign of multiplication is omitted in the indicated product; so when $(7+5)$ is to be multiplied by 2 we may write $2(7+5)$ or $(7+5)2$, the sign of multiplication being understood.

(1.) It is thus seen that *one of the uses of brackets is to indicate the order in which operations are to be performed.*

Thus, $10-(7-3)$ means that 3 is to be subtracted from 7 and the result is to be subtracted from 10.

If the values of the letters were given, what operations would you perform to find the values of :

$$a+(b+c),\quad a-(b+3),\quad a-(b-c),\quad (a-b)-(c-d)\ ?$$

The pupil should recognize that $3a^2$ is not the same as

$(3a)^2$. The latter means that a is first to be multiplied by 3 and the product is to be squared.

Thus, if $a=2$, $3a^2=3\times4=12$,
and $(3a)^2=3a\times3a=6\times6=36$.

(2.) *Brackets also indicate that the numbers within the brackets are to be considered as a single quantity, that is, they are used for the purpose of grouping.*

The dividing line between the numerator and denominator of a fraction has the same value as a pair of brackets.

Thus, in $\frac{a+b}{c+d}$, $a+b$ is a single quantity and so is $c+d$. The fractional form is another way of writing $(a+b)\div(c+d)$.

EXERCISE 5 (1-18, Oral)

Perform the operations indicated:

1. $10-(6+3)$.
2. $8-(4-2)$.
3. $15-6\div3$.
4. $(15-6)\div3$.
5. $3(4+7-5)$.
6. $(10+2)(5-1)$.
7. $10+2\times5-1$.
8. $(16+12)\div(6-2)$.
9. $7x-(8x-4x)$.
10. $(6a-2a)-(7a-4a)$.
11. $(3x+4x)\div7$.
12. $(10a-6a)\div2a$.
13. $(3x+9x)\div(6x-3x)$.
14. $3(7-5)-2(8-6)$.
15. $43x-(7x-4x)+2x$.
16. $x-(4y+3y-7y)$.
17. $(5b-4b)(3z-2z)$.
18. $\frac{6a-(7a-3a)}{8a-(6a+a)}$.

Indicate, using brackets:

19. That x is to be added to the sum of p and q.

20. That the sum of x and y is to be added to m.

21. That the sum of a and b is to be multiplied by 2.

22. That the difference of m and n, where m is greater than n, is to be subtracted from a.

23. If p is greater than q, that the difference of p and q is to be divided by the sum of m and n.

If $a=10$, $b=3$, $c=2$, find the value of:

24.* $8a-(2b+c)-5(a-b)$.

25. $7(a-b-c)-3(a-2b+c)$.

26. $(3a+2b-c)(a-3b)$.

27. $a^2+b^2+c^2-2(ab+bc+ca)$.

28. $\frac{a+3b-c}{2a-5b+2c}-\frac{2a-3b-3}{a+b-2c}$.

29. $\frac{a-b}{bc}+\frac{b-c}{c+a}-\frac{c+a}{a+2b}$.

30. When $a=6$ and $b=3$, show that

$$5(a-b)+3(a+b)=2(4a-b).$$

EXERCISE 6 (Review of Chapter I)

1. If x represents a certain number, what does $4x$ represent? $\frac{1}{2}x$? x^2? $3x^2$?

2. If a represents a number, what will represent 5 times the number? y times the number?

3. How do you indicate that y is to be added to x? That x is to be subtracted from y?

4. Indicate the sum of x and y diminished by a.

5. If one yard of cloth costs x cents, how many cents will 10 yards cost? How many dollars?

6. If a yard of ribbon is worth y cents, how much is a foot worth?

7. A man bought an article for x dollars and sold it at a loss of y dollars. What did he sell it for?

8. If I paid a dollars for b articles, how much did I pay for each? What would c articles cost at the same price?

9. A boy has a dollars. He earns b cents and then spends c cents. How many cents has he left?

10. I have x dollars. If I pay two debts of a dollars and b dollars, how much shall I have left?

11. If one number is x and another is 5 times as large, what is the sum of the numbers?

12. If one part of 10 is x, what is the other part?

13. A man worked m hours a day for 6 days. If he was paid $2 per hour, how much did he earn?

14. How far can a man walk in 5 hours at 4 miles per hour? In a hours at b miles per hour?

15. A man bought x acres of land at a dollars per acre and sold it at a loss of b dollars per acre. What did he sell it for?

16. What number is 15 greater than x? 15 less than x?

17. By how much does a^2 exceed b^2 when $a=7$, $b=3$?

Trinomial – one consisting of 3 terms
Binomial – one consisting of 2 terms
monomial – one consisting of 1 term

18. When $x=1$, $y=2$, $z=3$, what are the values of $x+y-z$, z $2x+5y-3z$, $7x-3y+z$?

19. If $a=3$, $b=4$, $c=0$, find the values of $2ab$, $4ac$, $a^2+b^2+c^2$, $5a^2-2b^2+4c^2$.

20. If $x=\frac{1}{2}$ and $y=\frac{1}{3}$, what are the values of $3x-2y$, $6xy$, $2x^2-3y^2$, $8x^3-27y^3$?

21. If $a=10$, $b=5$, $c=3$, find the values of $a(b+c)$, $a(b-c)$, $a(b^2-c^2)$, $c(a^2-b^2)$.

22. What is the sum of $2x$, $5x$, $7x$ and $3x$?

23. Simplify $5a-3a+11a+a-10a$.

24. What is the average of 20, 15, 0, 8, 12 ? Of $2a$, $3a$, $7a$? Of a, b, c, d ?

25. In 8 years a man will be x years old. How old was he 8 years ago ?

26. B has \$20 more than A, C has \$20 more than B. If A has \$$x$, how much has C ?

27. What is the sum of the numerical coefficients in the expression $3a+\frac{1}{2}ab+ac+\frac{1}{4}ad$?

28. Express 1000 as a power of 10; 32 as a power of 2; 81 as a power of 3; 64 as a power of 4.

29. Express 15, 105, $3ab$, $35x^2y$, as the products of simple factors.

30.* How long will it take me to walk a miles at 3 miles per hour and ride b miles at 12 miles per hour ?

31. A farmer buys 5 lb. of tea at x cents per lb. and 20 lb. of sugar at y cents per lb. He gives in exchange 7 lb. of butter at z cents per lb. If he still owes something, how much is it ?

32. If I buy 100 lb. of nails at a cents per lb. and 200 lb. at b cents per lb., what is the average cost per lb. ?

33. What is the total number of cents in x five-cent pieces, y ten-cent pieces and z half-dollars ?

34. What number is represented by $1000a+100b+10c+d$, when $a=1$, $b=2$, $c=3$, $d=4$? When $a=4$, $b=0$, $c=1$, $d=9$?

35. When $a=\cdot 2$ and $b=\cdot 1$, what are the values of $a+b$, ab, $\frac{a}{b}$, $\frac{a+b}{ab}$, a^2+b^2, a^2-b^2 ?

36. If A can do a piece of work in 10 days and B in 15 days, what fraction of the work can they together do in 1 day? What fraction if A could do it in x days and B in y days?

37. If $a=20$, $b=15$, $c=10$, $d=5$, find the difference between $(a+b)-(c+d)$ and $(a-b)-(c-d)$, also between $3(a+b)-5(c-d)$ and $5(a-d)-3(b-c)$.

38. When $x=7$ and $y=1$, the product of $x+y$, $x+2y$, $x-5y$ is how much greater than the product of $x-y$, $x-2y$, $x-3y$?

39. If $a=3$, find the value of $1+\cfrac{1}{1-\cfrac{1}{a+1}}$.

An equation [illegible]

[illegible]

[illegible]

Eg. $4x+5y-2=3x-4$

or $3x-5=0$

CHAPTER II S.K.T.

SIMPLE EQUATIONS

12. Idea of Equality. In weighing an article, when you see that the scales are balanced, what conclusion do you draw? If a 5 lb. bag of salt is placed in one scale pan, what weight (w) must be placed in the other pan to restore the balance? What must w be to balance a 3 lb. bag and a 4 lb. bag?

If the scales are balanced in each of the following figures, what must w be equal to?

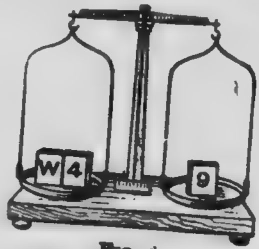


FIG. 1.

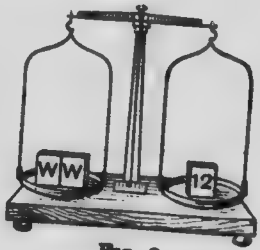


FIG. 2.

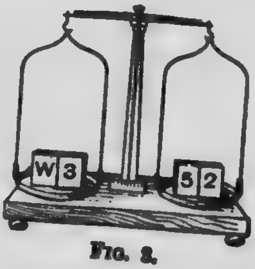


FIG. 3.

If $w+4=9$, as in fig. 1, what is w equal to?
If $w+w=12$, as in fig. 2, what is w equal to?
If $w+3=5+2$, as in fig. 3, what is w equal to?

If the scales are balanced and I add 2 lb. to one side, what else must I do to preserve the balance? What, if I take away 3 lb. from one side? If I double the weights on one side? If I halve the weights on one side?

13. The Equation. When a certain number is added to 10 the result is 27. What is the number?

The condition expressed in this problem might be more briefly shown in the form:

$$10+\text{a certain number}=27,$$

or in the form $10+?=27$, where the question mark stands for the required number.

Any other symbol would answer the same purpose as the

S.K.T.

question mark. Thus, if x represents the required number, then the problem states that

$$10+x=27.$$

This statement is called an **equation** and is merely a short way of stating what is given in the arithmetical problem preceding. In order that the statement may be true, it is easily seen that the symbol x must stand for the number 17.

Ex.—When a number is multiplied by 3, and 5 is subtracted from the product, the result is 19. What is the number ?

Here, if x stands for the number, the problem states that

$$3x-5=19.$$

Before the 5 was subtracted the product was evidently 5 more than 19 or $19+5$ or 24.

If 3 times the number is 24, then the number must be $\frac{1}{3}$ of 24 or 8.

The solution may be written more briefly thus :

If
$$3x-5=19,$$
$$\therefore\quad 3x=19+5=24,$$
$$\therefore\quad x=\tfrac{1}{3}\text{ of }24=8.$$

That 8 is the correct value for the number is shown by the fact, that when it is multiplied by 3 and 5 is subtracted from the product, the result is 19.

14. Solving an Equation. The process of finding the value of x, such that $3x-5=19$, is called "**solving the equation,**" and the value found for x is called the **root** of the equation.

EXERCISE 7 (Oral)

1. State the number for which the question mark stands in each of the following :

(1) $5+?=12$. (2) $?+12=20$. (3) $10-?=2$.
(4) $15=8+?$. (5) $40=62-?$. (6) $?-8=42$.

2. What is the number for which x stands in each of the following :

(1) $x+6=20$. (2) $8+x=32$. (3) $25=x+6$.
(4) $x-15=7$. (5) $10-x=8$. (6) $12=17-x$.

3. The first equation in Ex. 2 states that when a number is increased by 6 the result is 20. What does each of the other equations say?

4. If 3 times a number is 45, what is the number? If one-half of a number is 16, what is the number? If n stands for a given number, what would represent $\frac{1}{3}$ of the number? $\frac{3}{5}$ of the number?

5. For what number does n stand in each of the following equations:

(1) $4n=24$. (2) $\frac{1}{2}n=10$. (3) $\frac{3}{4}n=36$. (4) $\frac{7}{8}n=14$.

6. If $2x+5=11$, what is the value of $2x$? of x?

7. If $3m-2=13$, what is the value of $3m$? of m?

8. If $\frac{1}{3}p+3=10$, what is the value of $\frac{1}{3}p$? of p?

9. If $\frac{3}{5}x-11=7$, what is the value of $\frac{3}{5}x$? of x?

10. If $2(x+4)=14$, what is the value of $x+4$? of x?

Solve the equations:

11. $x+10=30$.
12. $3x-2=16$.
13. $5y+2=17$.
14. $4t-5=27$.
15. $2n=11$.
16. $7n-4=24$.
17. $3w+2=38$.
18. $\frac{1}{3}x-1=4$.
19. $2n+1=4$.
20. $3n-\frac{1}{2}=5\frac{1}{2}$.
21. $\frac{1}{3}w+2=5$.
22. $\frac{3}{4}x-5=15$.
23. $3(x+1)=30$.
24. $5(x-2)=45$.
25. $\frac{1}{2}(x-1)=3$.

15. **Axioms used in Solving Equations.** If two numbers are equal, what is the result when the same number is added to each?

Thus, if $x=6$, what is $x+2$ equal to?

What is the result when the same number is subtracted from two equal numbers; or when each is multiplied by the same number; or when each is divided by the same number?

Thus, if $x=10$, what is $x-4$ equal to? What is $3x$ equal to? What is $\frac{1}{2}x$ equal to?

The preceding conclusions may be stated thus:

(1) *If the same number be added to equal numbers, the sums are equal.*

(2) *If the same number be subtracted from equal numbers, the remainders are equal.*

(3) *If equal numbers be multiplied by the same number, the products are equal.*

(4) *If equal numbers be divided by the same number, the quotients are equal.*

These statements are called **axioms**, or self-evident truths, and are used in solving equations. The method is illustrated by the following examples :

Ex. 1.—Solve $3x-7=35$.

Add 7 to each side, $\therefore 3x-7+7=35+7$, axiom (1),
$\therefore 3x=42$.
Divide each side by 3, $\therefore x=\frac{42}{3}=14$, axiom (4).

Ex. 2.—Solve $\frac{1}{2}x+2=34$.

Subtract 2 from each side, $\therefore \frac{1}{2}x+2-2=34-2$, axiom (2),
$\therefore \frac{1}{2}x=32$.
Multiply each side by 2, $\therefore x=64$, axiom (3).

Ex. 3.—Solve $5x-3=2x+12$.

Add 3 to each side, $\therefore 5x=2x+15$.
Subtract $2x$ from each side, $\therefore 5x-2x=15$,
$\therefore 3x=15$.
Divide each side by 3, $\therefore x=5$.

The object of the changes which have been made in these equations is to get the quantities containing the unknown (x) to one side and the remaining quantities to the other side.

The unknown quantities are usually transferred to the left side, but sometimes it is better to transfer them to the right.

Ex. 4.—Solve $3m+20=5m-16$.

Add 16 to each side, $\therefore 3m+36=5m$,
Subtract $3m$ from each side, $\therefore 3m+36-3m=5m-3m$,
$\therefore 36=2m$,
$\therefore 18=m$ or $m=18$.

16. Verifying the Result. If we substitute 18 for m in the first side of the last equation we get

$$3m+20=3\times18+20=74.$$

If we substitute in the second side we get

$$5m-16=5\times18-16=74.$$

This process is called **verifying** or testing the correctness of the result. If the root obtained is the correct one, the two sides of the equation should be equal to the same number when the value found for the unknown is substituted.

The equation is then said to be **satisfied.**

The beginner is advised to verify the result in every case. Verify the results obtained in Ex.'s 1, 2 and 3.

EXERCISE 8 (Oral)

1. If $3x=15$, what does x equal ? What axiom is used ?

2. If $5x+2=17$, what does $5x$ equal ? What axiom is used ? What does x equal ? What axiom is used ?

3. If $2y-3=13$, what does y equal ? What two axioms are used ?

4. If $\frac{1}{3}x-4=6$, what does $\frac{1}{3}x$ equal ? What does x equal ? What two axioms are used ?

5. If $\frac{2}{3}x=6$, what does $\frac{1}{3}x$ equal ? What does x equal ? What two axioms are used ?

What is the value of x in the following equations :

6. $2x=18$. 7. $6x=72$. 8. $5x=16$. 9. $3x=6{\cdot}9$.
10. $x+20=25$. 11. $2x+1=15$. 12. $3x-1=20$. 13. $6x+5=29$.
14. $\frac{1}{3}x=8$. 15. $\frac{2}{3}x=12$. 16. $\frac{1}{4}x=2\frac{1}{4}$. 17. $\frac{5}{6}x=15$.

EXERCISE 9

Solve the following equations, giving full statements of the methods. In each case verify the result :

1. $3x+11=47$. 2. $2x+5=27$. 3. $4x-5=51$.
4. $3x-10=65$. 5. $4x=x+21$. 6. $4y=2y+80$.
7. $7x=60+3x$. 8. $\frac{1}{3}x+5=50$. 9. $6x+42=9x$.
10. $10x+3=3x+66$. 11. $6a-3a=a+5$. 12. $10x+20=20$.
13. $8m=36-4m$. 14. $20+6x+5=50-3x+11$.
15. $12x-652=7x+428$. 16. $764x-9=680x+12$.

17. Nine blocks of equal weights (w) together with a 20-gram weight are balanced by weights of 50 grams and 10 grams. Express this by an equation and find the weight of each block.

18. If $17x-11$ is equal in value to $5x+121$, what is the value of x ?

19. What value of y will make $11y+60$ equal to $20y-30$?

?

17. As we have already shown, an equation is merely *the statement in algebraic form of the condition given in an arithmetical problem.*

The solution of the problem is thus obtained by solving the equation.

EXERCISE 10

State the condition in each of the following problems in the form of an equation:

1. What must be added to 33 to make 50?
2. What must be taken from 90 to leave 40?
3. What is the number which when doubled is 36?
4. Five times a certain number is 45. What is the number?
5. If a number is doubled and 3 added, the result is 25. What is the number?
6. What number is doubled by adding 27?
7. What number is halved by subtracting 20?
8. If 8 is subtracted from $\frac{3}{4}$ of a certain number, the result is 7 What is the number?
9. Solve the equation in each of the preceding examples.

18. **Problems Solved by Equations.** The following examples will illustrate the method of solving problems by means of equations:

Ex. 1.—When I double a certain number and add 16, the result is 40. What is the number?

Let x represent the required number.
Then $2x$ is the double of the number.
Then $2x+16$ is the double with 16 added.
But the problem states that this is 40,

$$\begin{aligned} \therefore\quad 2x+16&=40,\\ \therefore\quad 2x&=24,\\ \therefore\quad x&=12. \end{aligned}$$

Therefore the required number is 12.

The result should be verified by showing that the number obtained satisfies the given problem.

Verification: When 12 is doubled I get 24 and when 16 is added I get 40. Therefore the result is correct.

Note that the substitution is made in the original problem, not in the equation. There might be an error in writing down the equation and then the solution obtained might satisfy the equation, but would not necessarily satisfy the given problem.

Ex. 2.—The number of pupils in a class is 33, and the number of boys is 7 greater than the number of girls. Find the number of each.

Let $x=$ the number of girls,

$\therefore\ x+7=$ the number of boys,

$\therefore\ x+x+7=$ the total number,

$\therefore\ x+x+7=33,$

$\therefore\ 2x=33-7=26,$

$\therefore\ x=13, \quad \therefore\ x+7=20,$

$\therefore$ the number of girls is 13 and the number of boys is 20.

Verification: $20+13=33,\ 20-13=7.$

Ex. 3.—Divide $100 among *A*, *B* and *C*, so that *B* may receive 3 times as much as *A*, and *C* $30 more than *B*.

Let $x=$ the number of dollars *A* receives,

$\therefore\ 3x=$ " " " *B* "

$\therefore\ 3x+30=$ " " " *C* "

$\therefore$ they all receive $(x+3x+3x+30)$ dollars,

$\therefore\ x+3x+3x+30=100,$

$\therefore\ 7x+30=100,$

$\therefore\ 7x=70,$

$\therefore\ x=10,$

$\therefore$ *A* receives $10, *B* $30 and *C* $60.

Verify this result.

19. Steps in the Solution of a Problem. The examples which have been given will show that in solving a problem the steps in the work are usually in the following order:

(1) *Read the problem carefully to see what quantity is to be found.*

(2) *Represent this unknown by a letter.*

(3) *If there be more than one quantity to be found, represent the others in terms of the same letter.*

(4) *Express the condition stated in the problem in the form of an equation.*

(5) *Solve the equation and draw the conclusion.*

(6) *Verify the solution by substitution in the problem.*

On referring to Ex. 1, we see that there was only one quantity to be found, and therefore step (3) did not appear in the solution. In Ex. 2 there were two quantities to be found, and when we represented the number of girls by x, we could represent the number of boys by $x+7$.

The pupil is advised to make full statements, in plain English, as to what the unknown represents.

Thus, in Ex. 3 to say, let $x=A$, or let $x=A$'s money, will only lead to difficulties.

NOTE.—The examples in the following exercise are to be solved by means of the equation and the results should be verified in every case. Although the answers to many of them may be given mentally, the pupil is advised to give complete solutions, so that he may become familiar with algebraic methods.

EXERCISE 11

1. If 37 is added to a certain number, the sum is 53. What is the number?

2. If 27 is subtracted from a number, the result is 5. What is the number?

3. A number was doubled and the result was increased by 27. If the sum is now 73, what was the number?

4. When a number is multiplied by 7, and 25 subtracted from the product, the result is 59. Find the number.

5. If five times a number be increased by 6, the sum is the same as if twice the number were increased by 15. Find the number.

6. What number if trebled and the result diminished by 36 gives twice the original number?

7. If you add 19 to a certain number the sum is the same as if you add 7 to twice the number. Find the number.

8. Five times a number, plus 19, equals nine times the number, minus 41. What is the number?

9. Two numbers differ by 11 and their sum is 51. Find the numbers.

10. The sum of two numbers is 47 and one exceeds the other by 15. What are the numbers ?

11. *A*'s salary is three times *B*'s and the difference of their salaries is $1500. Find the salary of each.

12. A horse and carriage are worth $360. The carriage is worth twice as much as the horse. Find the value of each.

13. Divide 93 into two parts so that one part will be 27 less than the other.

14. The length of a rectangle is three times the width. The perimeter is 72 feet. Find the sides.

15. *A* is twice as old as *B*. In 10 years the sum of their ages will be 41 years. What are their ages ?

16. Divide $500 between *A* and *B* so that *A* will receive $20 more than twice what *B* will receive.

17. The sum of two consecutive numbers is 59. What are the numbers ? (Let x be the smaller number, then $x+1$ will be the greater.)

18. Find three consecutive numbers whose sum is 150.

19. *A*'s age is twice *B*'s and *C* is 7 years older than *A*. The sum of their ages is 67 years. Find the age of each.

20. The difference between the length and width of a rectangle is 10 feet and the perimeter is 68 feet. Find the sides.

21. Divide $468 among *A*, *B* and *C*, so that *B* may get twice as much as *A*, and *C* three times as much as *B*.

22. A railway train travels m miles per hour. If it goes from Toronto to Montreal, a distance of 333 miles, in 9 hours 15 minutes, what is the value of m ?

23. A line 20 inches long is divided into two parts. The length of the longer part is ½ inch more than double the shorter one. Find the lengths of the parts.

24. What value of x will make $5x+6$ equal to $3x+40$?

25. If 5% of a sum is $48, what is the sum ?

26. An article sold for $2·61, the loss being 10%. What was the cost ?

27. Divide \$1496 among *A*, *B* and *C*, so that *B* will get three times *A*'s share and *C* will get \$100 more than *A* and *B* together.

28. *A* has five times as much money as *B*. After *A* has spent \$63 he has only twice as much as *B*. How much has *B* ?

29. If \$20 less than $\frac{1}{2}$ of a sum of money is \$10 more than $\frac{1}{3}$ of it what is the sum ?

30. Three boys sold 42 papers. The first sold $\frac{1}{2}$ as many as the third and the second sold $\frac{1}{4}$ as many as the third. How many did each sell ?

31. The sum of $\frac{1}{2}$ of a number and $\frac{1}{3}$ of the same number is 55. What is the number ?

32. A man paid a debt of \$4500 in 4 months, paying each month twice as much as the month before. How much did he pay the first month ?

33. The half, third and fourth parts of a certain number together make 52. Find the number.

34. Divide 72 into three parts so that the first part is $\frac{1}{2}$ of the second and $\frac{1}{3}$ of the third.

35. What number is that to which if you add its half and take away its third, the remainder will be 98 ?

36. If $3a=4bc$,

(1) Find a, when $b=10$, $c=15$.
(2) Find b, when $a=12$, $c=3$.
(3) Find c, when $a=8$, $b=\frac{1}{2}$.

EXERCISE 13 (Review of Chapter II)

1. State the four axioms which are used in solving equations.

2. Show that $x=18$ is the correct solution of the equation $3x-7=2x+11$.

3. Determine if 8 is a root of $3(x+6)=5(x-1)$.

4. Solve (*a*) $5x+3=2x+9$; (*b*) $1+2x=9-2x$; (*c*) $3x-7=8-2x$; (*d*) $7x+1=9x-9$; (*e*) $11x-1=5x+1$.

5. My house and lot cost \$16,800, the house costing five times as much as the lot. Find the cost of each.

6. A horse and carriage cost \$520. If the carriage cost \$60 more than the horse, what did the horse cost ?

7. Three farmers together raised 2700 bushels of wheat. A raised three times as much as B, and C raised twice as much as A. How much did each raise ?

8. What value of x will make $136-3x$ equal to $172-9x$?

9. Where r is the radius of a circle and c is the circumference, $c=2\pi r$, where $\pi=3\frac{1}{7}$.

(a) Find c, when $r=7$; when $r=42$.
(b) Find r, when $c=88$; when $c=11$.

10. If $s=\frac{1}{2}ft^2$, find s when $t=4$ and $f=32$; when $t=10$ and $f=32{\cdot}2$.

11. In a company of 98 persons, there are twice as many women as men, and twice as many children as women. How many children are there ?

12. Six boys and 15 men earn $264 a week. If each man earns four times as much as each boy, how much does a boy earn in a week ?

13. Five times a certain number, increased by 47 is equal to eight times the number, diminished by 43. What is the number ?

14. An agent charges 3 % commission for collecting an account. If his charge is $11·13, what was the amount of the account ?

15. Solve (a) $\cdot05x=4$; (b) $x+\cdot04x=208$; (c) $x-\cdot06x=235$; (d) $x+5\%x=630$.

16. If b is the base of a triangle and h is its height, the area (a) is given by the formula $a=\frac{1}{2}bh$.

(i) Find a, when $b=8$, $h=4$.
(ii) Find b, when $a=36$, $h=12$.
(iii) Find h, when $a=176$, $b=22$.

17. The sum of the unequal sides of a rectangle is 65 feet and their difference is 15 feet. Find the area of the rectangle.

18. If $6x-y=2x+y$, what is the value of y if $x=6$?

19. For what number does the question mark stand, if

$$5x+\tfrac{1}{3}=3x+\,?$$

is satisfied when $x=3$?

20. If 4 % of x together with 3 % of x is equal to 35, find x.

21. State a problem the condition of which is expressed by the equation $3x-20=x$.

22. B has $10 more than A, and C has $20 more than B. Together they have $190. How much has each ?

23. A turkey costs as much as three chickens. If 2 turkeys and 3 chickens cost $7·20, find the cost of a chicken.

24. What number increased by [illegible] of itself is equal to 60 ?

25. Divide \$6400 among A, B and C, so that B will get \$120 more than A, and C \$160 more than A.

26. The net income from an enterprise doubled each year for five years. If the total net income for the five years was \$7750, what was the income for the first year ?

27. If $2ab=3mn$,

(1) Find a, when $b=15$, $m=6$, $n=5$.
(2) Find b, when $a=12$, $m=2$, $n=2$.
(3) Find m, when $a=\frac{1}{2}$, $b=6$, $n=\frac{1}{2}$.
(4) Find n, when $a=\cdot 3$, $b=\cdot 6$, $m=\cdot 12$.

28. Show that 6 is a root of the equation

$$2(x-1)(x+2)=4(x+3)(x-5)+(x-2)(x+5).$$

29. The area of the United States is 4000 square miles more than seventy times the area of England. If the area of the United States is 3,560,000 square miles, find the area of England.

30 Solve and verify :

(1) $6850+x=27x+350$.
(2) $\frac{1}{2}x+\frac{1}{3}x+\frac{1}{4}x=3380$.
(3) $1607x+20=1762x-11$.

9

36

37

333

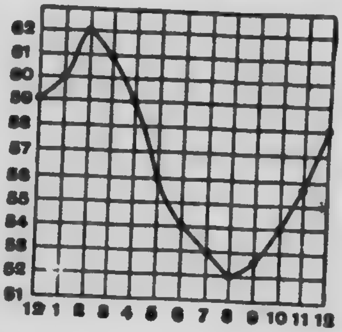
16

$\frac{1}{2} \times 32.2$

CHAPTER III

POSITIVE AND NEGATIVE NUMBERS

20. Arithmetical Numbers. In the diagram the hours from 12 noon to 12 midnight are represented on the horizontal line, and the temperature at each hour is shown by the position of a point on the corresponding vertical line.



Thus at 3 P.M. the temperature was 61°, at 7 P.M. 53° and at 11 P.M. 55·5°.

The points which show the temperature for each hour are connected by a curve. This curve gives a picture of the changes in temperature during these twelve hours.

These changes might be shown by a column of figures, but the curve exhibits the variations in temperature more readily to the eye. We can see at a glance when the temperature was rising and when falling, at what hours it was the same, that it rose or fell more rapidly during certain hours than during others.

Here we say that we have represented **graphically** the changes in temperature, and the curve shown is called a **graph.**

EXERCISE 18 (1-8, Oral)

Using the diagram, answer questions 1-8.

1. What was the temperature at 1 P.M., at 4 P.M., at 10 P.M.?
2. At what hours was the temperature the same?

3. What was the highest temperature? What the lowest?

4. What was the range of temperature?

5. Between what hours was it rising?

6. How much did it rise between 10 and 11? How much did it fall between 6 and 7?

7. When was it 60°, 58°, 55°?

8. Between what hours did it rise most rapidly? When did it fall most rapidly?

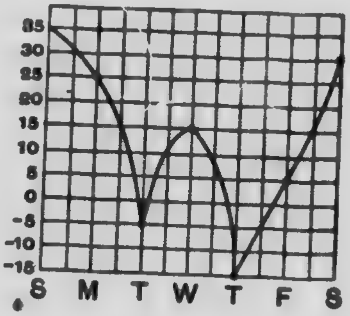
9. The percentage of games won by a baseball team, up to the beginning of each month of the playing season, was as follows:

June, 66; July, 63; Aug., 60·5; Sept., 62; Oct., 61·5.

Draw a graph showing these changes.

10. A boy's height in inches, for each year from the age of 7 to the age of 14, was 44, 47, 50, 51, 52·5, 54, 56·5, 58. Draw a graph to illustrate the variations in his height.

21. Negative Numbers. This diagram shows the average temperature for a week.



Thus on Monday it was 25° above zero, while on Thursday it was 15° below zero.

We might express this algebraically by saying that on Monday the temperature was $+25°$, and on Thursday it was $-15°$.

The number $+25$ is called a **positive** number and is read "positive" 25 or "plus" 25, while -25 is called a **negative** number, and is read "negative" 25 or "minus" 25.

A negative number is therefore one which is measured on the **opposite** side of zero from a positive number.

EXERCISE 14

1. Using algebraic signs, write down the temperature for each day in the diagram. Also read the temperature.

2. On what days was the temperature negative?

3. How much higher was it on Monday than on Thursday? How much lower on Tuesday than on Saturday?

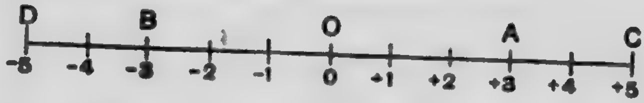
4. If the temperature is $-30°$ and it rises $40°$, how much will it be then? If it had fallen $10°$, how much would it have been then?

5. The temperature at which mercury freezes is $-39°$C. What does that mean? How much lower is it than the normal temperature of the blood which is $+37°$C.?

6. If the price at which a certain stock sells above par is positive and the price below par is negative, make a diagram similar to the preceding, showing the prices of a certain stock for a week, when the record was as follows:

Mon., 4 above par; Tues., 2 below; Wed., 1 above,
Thurs., at par; Fri., 3 below; Sat., $1\frac{1}{2}$ below.

22. Distances measured on a Horizontal Line.



On this diagram the distance between each successive marking represents one foot.

What is the length of OA? of OB? In what respect does OA differ from OB? How might we use signs to show this difference?

It is usual to consider measurements made to the right as positive and to the left as negative.

What point is $+5$ feet from O? What one is -5 feet from O? If a point moves from O, 4 feet to the right and then 7 feet to the left, how far is it then from O? Is the distance positive or negative?

We thus see that in addition to the numbers of arithmetic which begin with zero and extend indefinitely in one direction, we now have another series of numbers which also begin with zero and extend indefinitely in the **opposite direction.** In each series all integral and fractional numbers are included.

23. Further Examples of Negative Numbers.

(1) A man has property worth \$100, and debts amounting

to \$60. When he has paid his debts he will have property worth \$40.

Thus, $$\$100-\$60=\$40.$$

If, however, he has debts amounting to \$100, when these are paid he will have nothing left.

Thus, $$\$100-\$100=\$0.$$

If he has debts amounting to \$140, when he has paid all he can he will still owe \$40. We express this algebraically thus: $$\$100-\$140=-\$40.$$

In the first case we say that his net assets are \$40, in the second they are zero, and in the third they are *minus* \$40. When we say his assets are $-\$40$, we mean he is \$40 in debt.

It will be seen that the difference in meaning between $+40$ and -40 when referring to dollars is practically the same as the difference between $+40$ and -40, when referring to degrees of temperature, as in art. 21, or to distances measured in opposite directions on a horizontal line, as in art. 22.

(2) If a man gains \$20 on one transaction and loses \$15 on another, what is the net result? If he had lost \$25 on the second transaction what would have been the net result?

If we attach a plus sign to the result when it is a gain, how may we indicate a loss?

If G represents a sum gained and L a sum lost, state the result in each of the following, attaching the proper sign:

1. $\$30\,G+\$20\,G$. 2. $\$30\,G+\$20\,L$. 3. $\$30\,L+\$20\,L$.
4. $\$30\,L+\$20\,G$. 5. $\$40\,G+\$40\,L$. 6. $\$20\,G+\$60\,L$.

(3) If a game won is represented by $+1$, then -1 would represent a game lost.

In a series of games I find that my record is: won, lost, lost, won, lost, won, lost, won, won.

This might be represented thus:

$$+1-1-1+1-1+1-1+1+1=+5-4=+1.$$

What does this result mean?

Write in a similar way the following record: lost, lost, won, lost, won, lost, lost, won. Also the following: won, lost, drawn, won, won.

(4) In locating points on the earth's surface, the distance in degrees north of the equator (north latitude) is said to be positive, and south of the equator negative.

Thus, the latitude of Toronto is +44° and of Rio de Janeiro is −23°. What is the distance in degrees of latitude between these two cities?

The preceding illustrations show that a positive number differs from a negative number in **direction or quality.**

Thus, if +10 means 10 yards measured to the right; or 10° east longitude; or 10 games won; or 10 miles a boat goes up stream; or 10 minutes a clock is fast; or $10 in my bank balance; or 10 pounds lifted by a balloon; what would −10 mean in the corresponding cases?

24. Signs of Operation and Signs of Quality. The numbers +25 and −25 are the same in magnitude, but differ in direction or quality.

When a number is preceded by the sign +, it means that the number is taken in the positive direction or sense, and when preceded by the sign −, that it is taken in the negative direction.

It will thus be seen that we use the signs + and − with two different significations. When they are used to indicate the operations of addition or subtraction, they are called **signs of operation.** When they are used to indicate direction or quality, they are sometimes called **signs of quality.**

The beginner might think that this ambiguity would lead to confusion, but he will find that such is not the case.

When we read a quantity like −25, we should say "negative 25," but this is not followed in practice, as it is usually read "minus 25."

When no sign precedes a number, it is understood to be a positive number.

25. Absolute Value. The absolute value of a number is its value without regard to sign.

Thus, +8 and −8 have the same absolute value.

EXERCISE 15 (1-15, Oral)

1. What is the net property of a man who, (*a*) has $60 and owes $47, (*b*) has $40 and owes $50, (*c*) has $65 and owes $65 ?

2. What is the value of, (*a*) $40 − $30, (*b*) $40 − $60, (*c*) $30 − $20, (*d*) $20 − $30, (*e*) $10 − $0, (*f*) $0 − $10 ?

3. The temperature was −10° at 6 P.M. and 4° at 10 P.M. How many degrees did it rise in the interval ?

4. A liquid whose temperature is 20° is cooled through 30°. What is the final temperature ?

5. A vessel sailed on a meridian from latitude 15° to latitude −5°. How many degrees did it sail and in what direction ?

6. What is the distance between a place 90 miles due east of Toronto and another 60 miles due west ?

7. I am overdrawn at the bank $20. What must I deposit to make my balance $100 ?

If $-20+x=100$, what is x ?

8. What would a negative number mean in stating the height of a tree above the window of a house ? The height above sea level of the bottom of a well ?

9. A man buys a horse for $100 and sells him for $80. What is his *gain* and his *gain* % ?

10. A man travels 20 miles from *A* and his friend travels −10 miles from *A*. How far are they apart ?

11. What is the rise in temperature from −30° to −10° ?

If $-30+x=-10$, what is x ?

12. What is the distance between two places which are a miles and b miles west of Montreal, (1) if a is greater than b, (2) if b is greater than a ?

13. Denoting a date A.D. by + and B.C. by −, state the number of years between these pairs of dates :

(1) +1815 to +1915. (2) −20 to +75. (3) −65 to −37.
(4) −120 to +60. (5) −200 to +200. (6) +1900 to +1800.

14. Augustus was Roman Emperor from −31 to +14. How many years was he Emperor ? What is the difference between 14 and −31 ?

15. The First Punic War lasted from -264 to -241. How long did it last? What is the difference between -241 and -264?

16. A boy adds 15 marbles to his supply, gives away 10, buys 5 and gives away 12. How many has he thus *added* to his supply?

17. I have \$$a$ in the bank. If I issue a cheque for \$$b$, what is my balance when the cheque is paid? If $a=40$ and $b=50$, how do you interpret the result?

18. A has \$50 and B has \$20. A owes B \$10 and B owes A \$40. How much will each have when his debts are paid?

19. The weights of two pieces of iron are 65 lb. and 147 lb. If they are attached to a balloon with an upward pull of 239 lb., how would you represent the combined weight?

20. Represent graphically the following changes in the price of a stock:

Month.	July.	Aug.	Sep.	Oct.	Nov.	Dec.	Jan.	Feb.	Mar.	Apr.
Amount above par ...	6	1		4		5	2			3
Amount below par ...			2		3			1	4	

EXERCISE 16 (Review of Chapter III)

1. Using signs, express the results of the following transactions:

(1) A gain of \$10 followed by a loss of \$15.
(2) A loss of \$12 followed by a loss of \$4.
(3) A loss of \$8 followed by a gain of \$10.

2. What is the difference between 40° and $-3°$?

3. If an upward force or pull is positive and a downward force is negative, what single force is equal in effect to these pairs of forces:

(1) 10 lb., -3 lb. (2) 8 lb., -12 lb. (3) -7 lb., -2 lb.
(4) -9 lb., 3 lb. (5) 6 lb., -6 lb. (6) $2a$ lb., $-a$ lb.?

4. In firing at a target each hit counts 5 and each miss -3. If I fire 10 times and make 6 hits, what is my score? If I make only 2 hits what is my score?

5. What is the fall in temperature from 27° to $-11°$?
If $27-x=-11$, what is x?

6. In a 100 yards handicap race *A* has 3 yards start and *B* has −3 yards start. What do these mean? How far has each to run?

7. In solving a problem in which it is required to find in how many years *A* will be twice as old as *B*, I get the answer −10. What does this answer mean?

8. Find the average noon temperature for a week in which the noon temperatures were: 20°, 10°, 15°, 0°, 4°, −6°, −15°.

9. A train was due at 10 minutes to 3. How many minutes *before* three did it arrive if it was half an hour late?

10. A man travels 8 miles, then −6 miles, then 4 miles, then −11 miles. How far has he travelled? How far is he from the starting point and in what direction (positive or negative) from it?

11. Egypt was a Roman province from −30 to 616. How many years was this? What is the difference between 616 and −30?

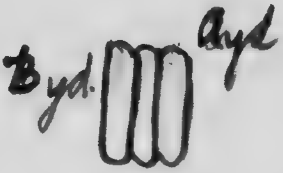
12. The daily average temperature for 14 days were: 6°, 5°, 0°, −4°, 2°, −6°, −2°, 0°, 5°, 1°, −1°, −6°, −3°, 3°. Show these variations by means of a graph.

13. If a gain of a dollar be the positive unit, what will represent a loss of $3·50?

14. The record of a patient's temperature for each hour beginning at 12 noon was: 100°, 100·5°, 102°, 101°, 104°, 101·5°, 99·5°, 98°, 97·5°, 97°. Represent these changes graphically, taking two spaces on the vertical line to represent one degree.

15. If the normal temperature of the body is 98·5°, write the record in the preceding question using positive and negative signs.

16. The minimum temperatures for the first 15 days of December were: 26°, 22°, 14°, 25°, 21°, 18·5°, 13°, 7·5°, 11°, 6°, −4°, −6°, −1°, 10°, 12·5°. Make a chart to show these variations.



97
103
15

CHAPTER IV

ADDITION AND SUBTRACTION

26. Addition of Positive Quantities. What is the result of combining:

(1) A gain of \$20 with another gain of \$10?

(2) A measurement of 5 feet to the right with another of 3 feet to the right?

(3) A rise in temperature of 10° with a rise of 8°?

(4) 6 points won with 4 points won?

As explained in Chapter III., we will consider all of these to be positive quantities, and we might show this by attaching the positive sign to each.

We might write these four questions as problems in addition, thus:

$$\begin{array}{cccc} +\$20 & +5\text{ feet} & +10^\circ & +\ 6\text{ points} \\ +\$10 & +3\text{ feet} & +\ 8^\circ & +\ 4\text{ points} \\ \hline +\$30 & +8\text{ feet} & +18^\circ & +10\text{ points} \end{array}$$

Similarly, the sum of $6a$ and $4a$ is $10a$, and the sum of $2x^2$, $5x^2$ and $6x^2$ is $13x^2$.

Here we have not prefixed any sign, and when that is the case the positive sign is understood.

We see then that *the sum of any number of positive quantities is always positive.*

27. Addition of Negative Quantities. We might change the data of the four questions in the preceding article so that all the quantities would be negative.

Thus, the first might be changed to —. "What is the result of combining a *loss* of $20 with a *loss* of $10 ?"

Read the other three questions making similar changes. What would now be the answer to each question ?

As problems in addition they would now appear thus :

$$\begin{array}{r} -\$20 \\ -\$10 \\ \hline -\$30 \end{array} \qquad \begin{array}{r} -5 \text{ feet} \\ -3 \text{ feet} \\ \hline -8 \text{ feet} \end{array} \qquad \begin{array}{r} -10^\circ \\ -\ 8^\circ \\ \hline -18^\circ \end{array} \qquad \begin{array}{r} -\ 6 \text{ points} \\ -\ 4 \text{ points} \\ \hline -10 \text{ points} \end{array}$$

Similarly, the sum of $-7x$ and $-5x$ is $-12x$, and the sum of $-5a^3$, $-2a^3$, $-a^3$ and $-6a^3$ is $-14a^3$.

Thus, *the sum of any number of negative quantities is negative, and is found by adding their absolute values and prefixing the negative sign to the result.*

EXERCISE 17 (Oral)

State the results of the following additions :

1. $\begin{array}{r} +\$3 \\ +\$5 \\ \hline \end{array}$ 2. $\begin{array}{r} -\$10 \\ -\$\ 8 \\ \hline \end{array}$ 3. $\begin{array}{r} -12^\circ \\ -10^\circ \\ \hline \end{array}$ 4. $\begin{array}{r} 3 \text{ yd.} \\ 5 \text{ yd.} \\ \hline \end{array}$

5. $\begin{array}{r} -\ 7 \\ -13 \\ \hline \end{array}$ 6. $\begin{array}{r} 4a^2 \\ 6a^2 \\ \hline \end{array}$ 7. $\begin{array}{r} -10x^2y \\ -\ 5x^2y \\ \hline \end{array}$ 8. $\begin{array}{r} -\frac{1}{2}abc \\ -\frac{3}{2}abc \\ \hline \end{array}$

9. Add $10x$, $12x$, $15x$.
10. Add $-3m^2$, $-m^2$, $-7m^2$.
11. Add $\frac{1}{8}m$, $\frac{3}{8}m$, $\frac{5}{8}m$, $\frac{7}{8}m$.
12. Add $-12y$, $-3y$, $-17y$.

28. Compound Expressions. An expression of one term is frequently called a simple expression, while one of more than one term is called a compound expression.

Thus, $2a$, $3x^2y$, $\frac{1}{4}abc$ are simple expressions, and $2a+3b$, $5x-3m+a^2$ are compound expressions.

29. Addition of Compound Expressions. In arithmetic if we wish to add two or more compound expressions, we write them under each other, with the like denominations in the same column.

We proceed in a similar way in algebra, writing like terms in the same column.

In arithmetic.	*In algebra.*
2 yd. 1 ft. 6 in.	$2a+b+6c$
3 yd. 1 ft. 4 in.	$3a+b+4c$
5 yd. 2 ft. 10 in.	$5a+2b+10c$.

If the like terms are not in the same order, they must be properly arranged for addition.

Ex.—Add $5x+3y-2z$, $4y-5z+3x$, $-3z+4x+y$.

Here the problem might be written thus :

$$\begin{array}{r} 5x+3y-\ 2z \\ 3x+4y-\ 5z \\ 4x+\ y-\ 3z \\ \hline \text{Sum}=12x+8y-10z. \end{array}$$

EXERCISE 18 (1-6, Oral)

Add :

1. 3 ft. 2 in.
   5 ft. 3 in.

2. $3x+2y$
   $5x+3y$

3. $5a-11b$
   $2a-\ 3b$

4. 6 hr. 10 min. 11 sec.
   5 hr. 12 min. 3 sec.
   2 hr. 15 min. 20 sec.

5. $5a-3b+2c$
   $2a-4b+3c$
   $5a-\ b+\ c$

6. $-\ a-3b+7c$
   $-3a-\ b+4c$
   $-5a-2b+\ c$

7.* $a+b-c$, $2b-3c+a$, $3b+5a-11c$.

8. $5x^2-7x+6$, $3-5x+x^2$, $-2x+4x^2$.

9. $2a-3b$, $3a-2b$, $4a$, $-b$, $a-b$.

10. $a+2b$, $b+2c$, $c+2a$, $a+b+c$.

11. $a-2b+c-3d$, $c-5b-d+2a$, $-b+c-d+a$.

12. $5x-3y$, $-2y+z$, $4z-y+3x$.

13. $\frac{1}{2}a-\frac{1}{3}b$, $\frac{2}{3}a-\frac{3}{5}b$, $\frac{3}{4}a-\frac{4}{5}b$, $\frac{7}{8}a-\frac{5}{8}b$.

14. $a^2+2b^2-3c^2$, $5b^2-c^2+2a^2$, $3a^2+b^2-2c^2$.

15. $a+b+c$, $b+c+d$, $c+d+a$, $d+a+b$.

16. $\frac{1}{2}x+\frac{1}{4}y-\frac{1}{3}z$, $\frac{1}{3}x+\frac{1}{4}y-\frac{2}{3}z$, $x+y-\frac{3}{4}z$.

30. Addition of Quantities with Unlike Signs.

What is the result of combining:

(1) A gain of \$20 with a loss of \$10?
(2) A gain of \$5 with a loss of \$15?
(3) A loss of \$8 with a gain of \$6?
(4) A loss of \$7 with a gain of \$12?

These might be written as problems in addition, thus:

$$\begin{array}{r} +\$20 \\ -\$10 \\ \hline +\$10 \end{array} \qquad \begin{array}{r} +\$\ 5 \\ -\$15 \\ \hline -\$10 \end{array} \qquad \begin{array}{r} -\$8 \\ +\$6 \\ \hline -\$2 \end{array} \qquad \begin{array}{r} -\$\ 7 \\ +\$12 \\ \hline +\$\ 5 \end{array}$$

It is thus seen, that when we add two quantities differing in sign, the sum is sometimes positive and sometimes negative. When is it positive and when is it negative?

How is the numerical part of the sum found when the signs are *alike*? How is it found when the signs are *different*?

The answers to these questions might be combined into the following rule:

When the signs are alike, the sum is found by arithmetical addition, and the common sign is affixed; when the signs are different, the sum is found by arithmetical subtraction, and the sign of the greater is affixed.

Ex. 1.—Find the sum of 6 and -8.

Here the result is -2, since the difference between 8 and 6 is 2 and the one with the greater absolute value is negative.

If there is doubt in any case, it is advisable to make the problem concrete by substituting for $+6$, a gain of \$6 and for -8, a loss of \$8, when the result will at once be evident.

Ex. 2.—Find the sum of $5a$, $-8a$, $-7a$, $6a$, $-2a$.

The sum of the positive quantities is $11a$.
The sum of the negative quantities is $-17a$.
The sum of $11a$ and $-17a$ is $-6a$,
$\therefore$ the required sum is $-6a$.

They might also be added in the order in which they come. Thus, the sum of $5a$ and $-8a$ is $-3a$, of $-3a$ and $-7a$ is $-10a$ of $-10a$ and $6a$ is $-4a$, of $-4a$ and $-2a$ is $-6a$.

Ex. 3.—Add $3a-11b+5c$, $6b-5a$, $5b-c+a$.

Write the expressions in columns as already explained.

$$\begin{array}{rrr|l} & & & a=b=c=1 \\ 3a & -11b & +5c & =-3 \\ -5a & +6b & & =+1 \\ a & +5b & -c & =+5 \\ \hline -a & & +4c & =+3 \end{array}$$

The sum is $-a+4c$ or $4c-a$, the sum of the second column being zero.

We may check the result by substituting particular numbers for the letters. Thus, if we substitute unity for each letter the first quantity becomes $3-11+5$ or -3, the second is $+1$, the third is $+5$, and the sum $(-a+4c)$ is $+3$. Since the sum of -3, $+1$ and $+5$ is $+3$, we assume that the work is correct.

EXERCISE 19 (1-12, Oral)

Add:

1. $\begin{array}{r} +6\text{ ft.} \\ -3\text{ ft.} \\ \hline \end{array}$ 2. $\begin{array}{r} -\$10 \\ +\$27 \\ \hline \end{array}$ 3. $\begin{array}{r} +10\text{ lb.} \\ -15\text{ lb.} \\ \hline \end{array}$ 4. $\begin{array}{r} -50^\circ \\ +40^\circ \\ \hline \end{array}$

5. $\begin{array}{r} -3 \\ 7 \\ \hline \end{array}$ 6. $\begin{array}{r} 5a^2 \\ -3a^2 \\ \hline \end{array}$ 7. $\begin{array}{r} -7x \\ 13x \\ \hline \end{array}$ 8. $\begin{array}{r} -8ab \\ 8ab \\ \hline \end{array}$

9. $\begin{array}{r} 3a-2b \\ 5a+3b \\ \hline \end{array}$ 10. $\begin{array}{r} 2x-7y \\ -3x+2y \\ \hline \end{array}$ 11. $\begin{array}{r} a+6b \\ a-6b \\ \hline \end{array}$ 12. $\begin{array}{r} 3a+2b-c \\ 2a-2b+c \\ \hline \end{array}$

13. $3x+5y$, $2x-3y$, $-4x-y$, $6x-4y$. (Check.)
14. $5m$, $-6m$, $-7m$, $8m$, $-9m$, $10m$.
15. $2a+3b-5c$, $6a-4b+c$, $3a+2b+4c$. (Check.)
16. $3a-5b$, $4b-3c$, $4c-3a$, $a+b+c$. (Check.)
17. $\frac{1}{2}x+y-\frac{1}{3}z$, $\frac{2}{3}x-\frac{1}{4}y+\frac{1}{2}z$, $x+\frac{1}{3}y+z$. (Check.)
18.* $a-2b+3c$, $b-2c+3d$, $c-3d+2a$, $b-2c-3a$.
19. $3x+5y-2z$, $2x-3y+4z$, $4x+y-5z$, $6x+2y+3z$.
20. $3ab-4ac+5bc$, $5ac-4bc-2ab$, $3bc-ab-ac$.
21. $6a^2-5ab+b^2$, $3a^2+7ab-2b^2$, a^2-ab+b^2.
22. $2x^2-3y^2+4z^2$, $5y^2-6z^2+x^2$, $2z^2+2y^2-3x^2$.
23. $\frac{1}{2}a-\frac{1}{3}b+\frac{1}{4}c$, $\frac{2}{3}a-\frac{3}{4}b-\frac{1}{2}c$.

24. If the sum of $13x-7$, $2x+5$ and $6-4x$ is 48, find the value of x and verify.

25. If the sum of $x-6$, $3x-6$ and $5x-6$ is the same as the sum of $12-x$, $12-3x$ and $12-5x$, find x and verify.

31. Indicated Additions. If we wish to use the sign of addition to indicate that b is to be added to a, we write it thus: $a+b$.

Similarly, if we wish to indicate the sum when -7 is added to 11, we write it $11+(-7)$, the negative quantity being enclosed in brackets.

To find the value of $11+(-7)$, we must add 11 and -7, which is done by subtracting their absolute values and prefixing the positive sign.

Thus, $$11+(-7)=11-7=4.$$
Similarly, $$6a+(-3a)=6a-3a=3a,$$
$$5m+(-3m)+(-m)=5m-3m-m=m,$$
and $$a+(-b)=a-b.$$

We thus see that, *to add a negative quantity is the same as to subtract a positive quantity of the same absolute value.*

If we wish to simplify a quantity like

$$(3a-2b)+(2a-3b),$$

we may write $2a-3b$ under $3a-2b$, and add in the usual way, or we may remove the brackets and say that the quantity

$$=3a-2b+2a-3b,$$
$$=5a-5b, \text{ when the like terms are collected.}$$

EXERCISE 20 (1-12, Oral)

Simplify:

1. $-3+4$.
2. $10+(-6)$.
3. $3+(-4)$.
4. $(-2)+(-3)$.
5. $5a+(-4a)$.
6. $7b+(-4b)$.
7. $-8a+(+7a)$.
8. $-5ab+(-2ab)$.
9. $9x^2+(-3x^2$
10. $-p+(-3p)$.
11. $(-3m)+(-8m)$.
12. $-a+(-2a)+(-3a)$.

13.* $10xy+(-3xy)+(-4xy)-xy+(-5xy)$.

14. $-b+(-2b)+(-3b)+(-4b)+10b$.

15. $-5p^2+3p^2+(-2p^2)+8p^2$.

16. $(2m+3n)+(5m-n)+(3m-5n)$.

17. $(6x+3y-4z)+(x+2y-z)+(y+z-7x)$.

18. $a+(-b)+b+(-c)+c+(-a)$.

19. $x+(a-2b+c)+(b-2c+a)+(c-2a+b)$.

20. When -20 is subtracted from 10, the difference is 30. Show that this is true by adding the difference to the quantity which was subtracted.

21. Show by addition, that when $2a-b+5c$ is subtracted from $3a-4b+3c$ the remainder is $a-3b-2c$.

22. Solve $(2x+3)+(3x-5)+(5x-1)=57$. (Verify.)

23. Solve $(8x-7)+(-4x-3)=(-5x-7)+(7x-2)$. (Verify.)

22. Subtraction is the Inverse of Addition. To subtract 4 from 7 is equivalent to finding the number which added to 4 will make 7.

Thus every problem in subtraction may be changed into a corresponding problem in addition.

If we wish to subtract -4 from 7, we enquire what number added to -4 will make 7. We might make the problem concrete by finding what must be added to a loss of \$4 to result in a gain of \$7, and the answer is evidently a gain of \$11.

$\therefore$ when -4 is subtracted from 7 the remainder is 11.

Thus, 7 less $-4=11$, because $-4+11=7$.
-10 less $-3=-7$, because $-3+(-7)=-10$.
$6b$ less $-4b=10b$, because $-4b+10b=6b$.

EXERCISE 21 (Oral)

What must be added to

1. A gain of \$10 to give a gain of \$15?
2. A gain of \$8 to give a gain of \$3?
3. A gain of \$5 to give a loss of \$4?

4. A loss of \$6 to give a gain of \$3 ?

5. A loss of \$20 to give a loss of \$15 ?

6. A loss [illegible] to give a loss of \$8 ?

7. 8 to give 12.
8. 10 to give 2.
9. -5 to give 2.
10. -8 to give -2.
11. $7x$ to give $3x$.
12. $3a$ to give $-5a$.
13. $5x^2$ to give $-2x^2$.
14. $-8x$ to give $-10x$.
15. $3abc$ to give $2abc$.
16. $-6y^2$ to give $-9y^2$.
17. $-5t$ to give $-4t$.
18. a to give $-a$.

33. Rule for Subtraction. Examine the following subtractions :

$$(1)\ \begin{array}{r} 9 \\ 3 \\ \hline 6 \end{array} \qquad (2)\ \begin{array}{r} 9 \\ -3 \\ \hline 12 \end{array} \qquad (3)\ \begin{array}{r} -9 \\ 3 \\ \hline -12 \end{array} \qquad (4)\ \begin{array}{r} -9 \\ -3 \\ \hline -6 \end{array}$$

In (1) the result 6 might have been found equally well by *adding* -3 to 9, in (2) by *adding* $+3$ to 9, in (3) by *adding* -3 to -9, and in (4) by *adding* $+3$ to -9.

Thus we see that the problems might have been re-written as problems in addition by changing the sign of the quantity to be subtracted.

We would then have these problems in addition :

$$(1)\ \begin{array}{r} 9 \\ -3 \\ \hline 6 \end{array} \qquad (2)\ \begin{array}{r} 9 \\ +3 \\ \hline 12 \end{array} \qquad (3)\ \begin{array}{r} -9 \\ -3 \\ \hline -12 \end{array} \qquad (4)\ \begin{array}{r} -9 \\ +3 \\ \hline -6 \end{array}$$

We have therefore the following rule for subtraction. *Change the sign of the quantity to be subtracted and add* as in algebra.

To subtract compound expressions, apply the rule to the like terms of the expressions.

Thus, to subtract $\begin{array}{r} 6a-3b+2c \\ 3a+4b-6c \\ \hline \end{array}$ change to $\begin{array}{r} 6a-3b+2c \\ -3a-4b+6c \\ \hline 3a-7b+8c \end{array}$ and add.

EXERCISE 22

Re-write the following problems in subtraction as problems in addition and find the result:

$$\begin{array}{llllllll}
1. & \begin{array}{r}25\\-13\\\hline\end{array} & 2. & \begin{array}{r}-11\\+\ 3\\\hline\end{array} & 3. & \begin{array}{r}-10a\\-\ 8a\\\hline\end{array} & 4. & \begin{array}{r}20x\\-\ 4x\\\hline\end{array}
\end{array}$$

$$\begin{array}{llllllll}
5. & \begin{array}{r}-16a^2\\-20a^2\\\hline\end{array} & 6. & \begin{array}{r}5b\\8b\\\hline\end{array} & 7. & \begin{array}{r}4x^2y\\-3x^2y\\\hline\end{array} & 8. & \begin{array}{r}-23m^2\\-\ 6m^2\\\hline\end{array}
\end{array}$$

$$\begin{array}{llllllll}
9. & \begin{array}{r}3a+7b\\2a-2b\\\hline\end{array} & 10. & \begin{array}{r}6x-3y\\-2x+4y\\\hline\end{array} & 11. & \begin{array}{r}5m+4n\\5m-4n\\\hline\end{array} & 12. & \begin{array}{r}6x^2-5x+2\\3x^2-2x-3\\\hline\end{array}
\end{array}$$

13 Subtract $a-2b+3c$ from $3a-4b+c$.

14. Subtract $5x^2-11x+4$ from $3x^2-2x+5$.

As soon as possible the pupil should learn to subtract, without actually changing the signs, but by making the change mentally.

EXERCISE 23 (1-8, Oral)

Subtract:

$$\begin{array}{llllllll}
1. & \begin{array}{r}8\\-3\\\hline\end{array} & 2. & \begin{array}{r}-9\\4\\\hline\end{array} & 3. & \begin{array}{r}-12\\-\ 6\\\hline\end{array} & 4. & \begin{array}{r}-7a\\2a\\\hline\end{array}
\end{array}$$

$$\begin{array}{llllllll}
5. & \begin{array}{r}-8x\\-3x\\\hline\end{array} & 6. & \begin{array}{r}-7m\\-2m\\\hline\end{array} & 7. & \begin{array}{r}-\ 4abc\\11abc\\\hline\end{array} & 8. & \begin{array}{r}-9\frac{1}{3}\\-1\frac{1}{3}\\\hline\end{array}
\end{array}$$

$$\begin{array}{llllllll}
9. & \begin{array}{r}3x+4y\\x-2y\\\hline\end{array} & 10. & \begin{array}{r}2a-3b\\3a+2b\\\hline\end{array} & 11. & \begin{array}{r}7x^2-5\\3x^2-7\\\hline\end{array} & 12. & \begin{array}{r}6m-3n\\-3m+6n\\\hline\end{array}
\end{array}$$

13. $2m+4n-3p$ from $5m-3n+6p$, and verify the work by addition.

14. Find the remainder when $6a-4b-5c$ is subtracted from $2a+3b-11c$. (Verify.)

15.* Subtract $5a-3b$ from the sum of $2a-b$ and $6a-4b$.

16. Subtract $a-2b+c$ from $6a-b+5c$ and from the remainder subtract $3a+b-c$.

17. Subtract the sum of $3x^2-5x+6$ and $5x^2+4x-3$ from $5x^2-x+3$. Check when $x=1$.

18. What must be added to $2m+3n-4p$ to give $5m-n-2p$? (Verify.)

19. By how much is $7a^2-15a-11$ greater than $3a^2-11a+4$?

20. By three subtractions simplify

$$(6a+10b-c)-(3a+4b-2c)-(a-3b+4c)-(2a+7b-3c).$$

21. Subtract the sum of $2p-5q-3r$, $-p+3q-2r$ and $4p+6q-4r$ from the sum of $3p-4q+5r$, $3q-4r+5p$ and $3r-4p+5q$.

22. Subtract $2a-3b+5c$ from zero.

23. What is the excess of 15 over 10? 8 over -2? -4 over -11? $a+b$ over $a-2b$? $3x^2-5x+2$ over $2x^2-11x+7$?

24. Add a^2+2a-5 to the excess of $2a^2-4a+3$ over $a^2-3a+10$.

25. Subtract the sum of $a-3b+c$, $b-3c+a$, and $c-3a+b$ from zero.

26. What must be added to the sum of x^2-5x, $2x^2-3x+4$ and $6x-3x^2$ so that the result will be unity?

27. From $2x-3x^2+5-x^3$ take $3-11x^3-5x^2-6x$.

34. Indicated Subtractions. If we wish to indicate that -3 is to be subtracted from 9 we write it: $9-(-3)$. From the preceding this is at once seen to be equal to $9+(+3)=12$.

Also,

$$-8-(-5)=-8+(+5)=-3.$$
$$-3a-(-5a)=-3a+5a=2a.$$
$$a-(-b)=a+(+b)=a+b.$$

We thus see that *to subtract a negative quantity is the same as to add a positive quantity of the same absolute value.*

Similarly, $(3a+4b)-(2a-3b)=(3a+4b)+(-2a+3b),$

$$=3a+4b-2a+3b,$$
$$=a+7b.$$

Also $\quad a-(b+c)=a+(-b-c)=a-b-c,$

and $\quad a-(b-c)=a+(-b+c)=a-b+c.$

Thus, *brackets which are preceded by a minus sign may be removed if the signs of all the quantities within the brackets be changed.*

Ex.—Simplify $5x-3y+4z-(3x-2y+2z)$.

We may consider this as an ordinary problem in subtraction and proceed in the usual way.

$$\begin{array}{r} 5x-3y+4z \\ 3x-2y+2z \\ \hline 2x-\ \ y+2z \end{array}$$

We may, however, remove the brackets using the rule and then collect the like terms, thus :

The expression $=5x-3y+4z-3x+2y-2z,$
$=2x-y+2z.$

EXERCISE 24 (1-10, Oral)

Simplify :

1. $10-(-3)$.
2. $-5-(-6)$.
3. $-7a-(-4a)$.
4. $8x-(-3x)$.
5. $(-2m)-(-3m)$.
6. $-(-b)+b$.
7. $8-(-4)-(-2)$
8. $8ab-10ab-(-7ab)$.
9. $m-(-3m)-(-5m)$.
10. $-4x^2+(-3x^2)-(-7x^2)$.
11.* $(5x-2y)-(2x-4y)$.
12. $3a-11b-(5a-8b)$.
13. $2a-3b+5c-(a-4b+5c)$.
14. $(a+b)+(2a-3b)-(4a-3b)$.
15. $a+b-c-(b+c-a)+(a+b-c)$.
16. $(6x^2-3x+5)+(2x^2-5x-6)-(5x^2-8x+2)$.
17. Find the value of $5a+b$, when $a=2$, $b=-3$.
18. Find the value of $2a+3b-c$, when $a=1$, $b=2$, $c=-3$.
19. By two different methods find the value of $a-(b-c)$, whe $a=20$, $b=10$, $c=7$.
20. Solve and verify :

    (1) $2x-3-(x-4)=8$.
    (2) $3x-1-(x-3)-(x+7)=40$.
    (3) $1-(4-x)-(5-x)-(6-x)=52$.
21. What value of x will make $5x-6$ exceed $3x-11$ by 70 ?
22. Find the value of $3x^2-2x+5-(2x^2+x-1)$, when $x=0, 1, 2, 3, 4$.

23. Remove the brackets and simplify :

(1) $(a+3b-11c)-(b+3c-8a)-(c+5a-2b)$.
(2) $(a-3b)-(b-3c)+(c-3d)-(d-3a)$.
(3) $-(3x-y+2z)-(2x-3y+4z)-(3y-6z-5x)$.
(4) $-(a-b)-(b-c)-(c-d)-(d-a)$.

EXERCISE 25 (Review of Chapter IV)

1.* Find the sum of $5a$, $-3a$, $7a$ and $-8a$.

2. Find the sum of four consecutive integers of which x is the least.

3. Find the sum of five consecutive integers of which n is the middle one.

4. Add $3a-2b+7c$, $5b-3c-2a$ and $c-a-3b$.

5. Subtract $-10b$ from $-6b$, $-3a$ from $5a$.

6. From $4a-3b+5c$ subtract $2a-5b-c$.

7. Subtract $5x-3y+4z$ from $4x-2y-z$.

8. What must be added to $3a-5b+6c$ to give $6a-7b+4c$?

9. If $x+y=10$ and $x-y=4$, what is the value of $2x$? What is the value of $2y$?

10. What is the sum of the coefficients in $6a-11b+c-3d$?

11. What must be added to $x-y$ to give 0 ?

12. When $x=3$ and $y=4$, what is the remainder when x^2-y^2 is subtracted from $2xy$?

13. What is the difference between $2a-b-c$ and $a-b+c$? Give two answers.

14. To the sum of $3m-4n$ and $2m-3n$ add the sum of $m+7n$ and $3m-2n$.

15. From $4x^2+3x-7$ subtract the sum of $2x^2+7x-5$ and $2x^2-8x+7$.

16. By how much is $3x-7$ greater than $2x+5$? For what value of x would they be equal ?

17. Simplify $a+(2b-3c)-(c+a)$.

18. If $x=a+2b-3c$, $y=b+2c-3a$ and $z=c+2b-3a$, find the value of (1) $x+(y+z)$, (2) $x-(y-z)$, (3) $x-(y+z)$.

19. From the sum of $\cdot5x+\cdot4y$ and $\cdot3x-\cdot2y$ subtract the sum of $\cdot2x-1\cdot1y$ and $\cdot4x+1\cdot3y$.

20. By how much is $3x^2-5x+11$ greater than $3x^2-8x+17$? What is the meaning of the result when $x=2$? when $x=1$?

21. From $\frac{3}{4}a-\frac{1}{3}b+\frac{1}{2}c$ take $\frac{1}{4}a-\frac{2}{3}b-\frac{1}{2}c$.

22. If $a+b+c=0$ when $a=3x-4y$ and $b=4y-5z$, what is the value of c?

23. Solve $5x-3-(x-4)-(x-2)=27$. (Verify.)

24. What value of x will make $3x-2$ exceed $x-7$ by 63?

25. When $a=1$, $b=2$, $c=3$, the sum of $x+a-3b+4c$, $2x+b-3c+4a$ and $3x+c-a-b$ is 124. Find the value of x.

26. Using brackets, indicate that the sum of a and b is to be diminished by the sum of c and d. If $a=2x-3$, $b=5-3x$, $c=3x-\frac{1}{2}$, $d=\frac{1}{2}-6x$, what is the result?

27. Solve $2-(x-\frac{1}{2})-(\frac{1}{4}-3x)=16\cdot25$.

28. Subtract $2m-7n-4x$ from zero.

29. What must be added to $a-(1-b)-(1-c)$ to produce unity?

30. Subtract the sum of $2a+3b-4c+d$ and $a-2b-c-d$ from the excess of $4a-b+c$ over $a+b+c$.

If the signs are the same the answer is plus if the signs are different. The signs are minus
ff

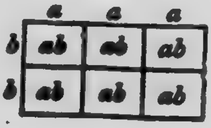
CHAPTER V

MULTIPLICATION AND DIVISION

35. Multiplication of Simple Positive Quantities. The factors of a product may be taken in any order.

Thus, $3\times5\times4=3\times4\times5=5\times4\times3=$ etc.
Also $a\times b\times c=a\times c\times b=b\times a\times c=$ etc.

The latter product may be written *abc*, *acb*, etc.



$$\begin{aligned}3a\times2b&=3\times a\times2\times b,\\&=3\times2\times a\times b,\\&=6ab.\end{aligned}$$

Make a diagram to show that $4x\times2y=8xy$.

Similarly,
$$\begin{aligned}3ab\times5cd&=3\times a\times b\times5\times c\times d,\\&=3\times5\times a\times b\times c\times d,\\&=15abcd.\end{aligned}$$

Thus, *the coefficient of the product is obtained by multiplying the coefficients of the factors, and the literal part of the product by multiplying the literal parts of the factors.*

36. The Index Law for Multiplication.

$$a^2\times a^3=a\times a\times a\times a\times a=a^5.$$
$$2x\times3x^2=6\times x\times x\times x=6x^3.$$
$$m^4\times m^3=m\,.\,m\,.\,m\,.\,m\times m\,.\,m\,.\,m=m^7.$$

Similarly, $3y^3\times4y^3=12y^6$,
and $x^3\times x^4\times x^5=x^{3+4+5}=x^{12}$.

Thus, *the index of the product of powers of the same quantity is found by adding the indices of the several factors.*

EXERCISE 36 (Oral)

Find the product of:

1. $2x$, $3y$.
2. $4m$, $5n$.
3. $\frac{1}{2}x$, $4y$.
4. $3x$, $4x$.
5. $6x$, $\frac{1}{3}x$.
6. $3ab$, $4xy$.
7. a^2, a.
8. $3y^2$, $2y^3$.
9. ab, ac.
10. $2x^2$, $4x^3$.
11. $5p^2$, $4p^4$.
12. $5x^2$, $3xy$
13. $(4x^2)^2$.
14. $3ax$, $2ax$.
15. t^2, t^3, t^4.
16. ab, ac, a.
17. $5a^2$, $3a$, 2.
18. $(4a)^3$.
19. $\frac{1}{2}a$, $\frac{1}{3}b$, $\frac{1}{4}c$.
20. $(3b)^2$, $(2b)^2$.
21. $\frac{2}{3}m$, $6n$. $5mn$.

37. Multiplication by a Negative Quantity.

4×3 is a short way of writing $4+4+4=12$.

-4×3 is a short way of writing $(-4)+(-4)+(-4)=-12$.

Hence multiplication by a positive integer means that the multiplicand is taken as an **addend** as many times as there are units in the multiplier.

Also, we shall *define* multiplication by a negative integer as meaning that the multiplicand is taken as a **subtrahend** as many times as there are units in the multiplier.

According to this definition then

$$4\times -3=-4-4-4=-12,$$
$$-4\times -3=-(-4)-(-4)-(-4),$$
$$=+4+4+4=+12.$$

We may state these results in algebraic symbols, thus:

$$(+a)\times(+b)=+ab,$$
$$(-a)\times(+b)=-ab,$$
$$(+a)\times(-b)=-ab,$$
$$(-a)\times(-b)=+ab.$$

38. Rule of Signs for Multiplication. Examine the preceding statements and state when the product has a *positive* sign, and when it has a *negative* sign.

The rule of signs for multiplication may be stated in the form:

The product of two factors with like signs is positive, and of two factors with unlike signs is negative

EXERCISE 27 (Oral)

State the product of:

1. $6, -7$. 2. $-4, 3$. 3. $-4, -5$.
4. $\frac{2}{3}, -5$. 5. $3x, -2$. 6. $-x, y$.
7. $-x, -2y$. 8. $2m, -3n$. 9. $-x, -3xy$
10. $-a(-b)$. 11. $(-6)^2$. 12. $-ay(-x)$.
13. $-2mv(-v)$. 14. $-\frac{1}{2}x, -12x$. 15. $-2x^2 \times -3x^2$
16. $x^2, -x$. 17. $-2a^2, -3a$. 18. $-ab \times -3cd$.
19. $-5xy, 2x^2$. 20. $-a^3, -a$. 21. $3a^2b \times -ab$.
22. $5x^2y^3, -xy^2$. 23. $3a^2b^3c^4, -ab^2c$. 24. $-x^2yz^3, 5x^2y^2z$.

25. Since $-4 \times 3 = -12$, $-4 \times 2 = -8$, $-4 \times 1 = -4$, what would you expect -4×0 to be equal to? Also -4×-1 and -4×-2?

39. Multiplication of several Simple Factors.

Ex. 1.—Multiply $2a, -3b, -4ab, -b$. $-24a^2b^3$

The product of $2a$ and $-3b$ is $-6ab$.
" " " $-6ab$ and $-4ab$ is $24a^2b^2$.
" " " $24a^2b^2$ and $-b$ is $-24a^2b^3$,
∴ the required product is $-24a^2b^3$.

Of course the factors may be multiplied in any order we choose. If we multiply all the negative factors first, what sign will their product have?

What sign will the product have if we multiply four negative factors? Five negative factors? Twenty negative factors?

The product will be *negative* when the number of negative factors is *odd* and will be *positive* when the number of negative factors is *even*. Any number of positive factors will evidently not affect the sign of the product.

Ex. 2.—Multiply $3x, -5xy, -6x^2, -y, 6y^3$. $-540x^4y^5$

(1) The sign of the product is $-$, since there is an odd number of negative factors.

(2) The numerical coefficient $= 3 \times 5 \times 6 \times 6 = 540$.

(3) The literal part of the product $= x \cdot xy \cdot x^2 \cdot y \cdot y^3 = x^4y^5$,
∴ the complete product $= -540x^4y^5$.

56
920
54

Ex. 3.—Find the values of $(-2)^3$, $(-2)^4$, $(-2)^5$.

In $(-2)^3$ the number of factors is odd,

$$\therefore (-2)^3 = -2^3 = -8.$$

Similarly, $(-2)^4 = 16$ and $(-2)^5 = -32$.

EXERCISE 23 (1-18 Oral)

Find the product of:

1. $3, 4, -5$.
2. $3, -4, -5$.
3. $-3, -4, -5$.
4. $-a, -b, -c$.
5. $2a, 3a, -4a$.
6. $-\frac{1}{2}, -\frac{1}{3}, 12$.
7. $3x, -2xy, -y$.
8. $-3, -3, -3$.
9. $(-4)^3$.
10. $-2, -3, -4, -5$.
11. $(-2)^2 \times (-3)^2$.
12. $-2x, -2x, -2x$
13. $-a, -2a, -3a, -4a, -5a$.
14. $2x, -3x, 4y, -y$.
15. $5xy, -3xy, -2x, -2y$.
16. $-1, -2, -3, -\frac{1}{2}, -\frac{1}{3}$.
17. What is the square of $-2a$, of $-3xy$, of $-4a^2bc$?
18. What is the cube of -5, of $-x$, of $-2x^3$?

19.* If $x=-1$ and $y=-2$, find the values of:

$$x^2, y^2, x^2+y^2, x^2-y^2, x^3, y^3, x^3+y^3, x^3-y^3.$$

20. Find the value of $3x^2+2x-5$ when $x=-2$; when $x=-3$; when $x=-4$.

21. Write without the brackets:

$$(-a)^2, (-a)^3, (-2)^3, (-1)^7, (-1)^8, (-3)^4, (-1)^3 \times (-2)^5.$$

22. Find the sum of the squares of $-2, -3, -4$. Find also the square of their sum.

23. Find the value of $(a-b)^2+(b-c)^2+(c-a)^2$ when $a=6$, $b=4$, $c=2$.

24. When $a=2$, $b=1$, $c=-3$, show that $a^3+b^3+c^3=3abc$.

25. When $x=3$ and $y=-2$, how much greater is $(x-y)^3$ than x^3-y^3?

If $a=-1$, $b=-2$, $c=-3$, $d=-4$, find the value of:

26. $3a+2b+c-4d$.
27. $a^2+b^2+c^2+d^2$.
28. $ab+ac+bc+cd$.
29. $a^2d^2-b^2c^2$.
30. $abc+bcd+cda+dab$.
31. $a^3+b^3+c^3+d^3$.

odd = –

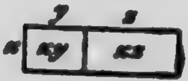
[illegible] = +

[illegible] Compound Multiplication. We multiply a compound quantity by a simple one in a manner similar to the method in arithmetic.

In arithmetic.		*In algebra.*
3yd. 1ft. 4in.	$23=2.10+3$	$3a+4b-\ 5c$
2	$2\quad 2$	2
6yd. 2ft. 8in.	$46=4.10+6$	$6a+8b-10c$

If we wish to indicate the product of x and $y+z$, we write it in the form $x(y+z)$, which we see is equal to $xy+xz$.

The diagram shows how this product may be illustrated geometrically.

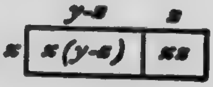


Make a similar diagram to show that

$$a(b+c+d)=ab+ac+ad.$$

Similarly, $x(y-z)=xy-xz$.

Can you see that the diagram is a geometrical illustration of this ?



Ex.—Simplify $3(a-b)-4(b-c)-2(a-b+c)$.

The expression $=(3a-3b)-(4b-4c)-(2a-2b+2c)$,
$=3a-3b-4b+4c-2a+2b-2c$,
$=a-5b+2c$.

EXERCISE 29

Copy and supply the products :

1. $2a+b$ by 4
2. $3a-2b$ by 7
3. $2m-5n$ by -6
4. $4x-3y$ by $2x$
5. $3x-4y$ by $-2y$
6. $2a+5b-c$ by $-a$

$$\begin{array}{lll} 1.\ \ \begin{array}{l} 2a+b \\ 4 \\ \hline \end{array} & 2.\ \ \begin{array}{l} 3a-2b \\ 7 \\ \hline \end{array} & 3.\ \ \begin{array}{l} 2m-5n \\ -6 \\ \hline \end{array} \\ 4.\ \ \begin{array}{l} 4x-3y \\ 2x \\ \hline \end{array} & 5.\ \ \begin{array}{l} 3x-4y \\ -2y \\ \hline \end{array} & 6.\ \ \begin{array}{l} 2a+5b-c \\ -a \\ \hline \end{array} \end{array}$$

7. $2(3x-11)$.
8. $3(5x-2y)$.
9. $-2(3x-y)$.
10. $3x(x^2+5x-2)$.
11. $5xy(2x^2-xy)$.
12. $3mp(5-mp)$.

Simplify :

13.* $3(a+b)+4(b+c)+5(c+a)$.

14. $2(x-2y)+3(x-y)-(4x-3y)$.

15. $3(2m-3n)-5(m-n)+2(m+2n)$.

16. $4(a-2b+c)-3(b-2c+a)-2(5c-4a-5b)$

17. $\frac{1}{3}(2a-3b)+\frac{1}{4}(2a+5b)+\frac{1}{6}(5a+b)$.

18. $x(x-1)+2x(x-3)+3x(x+5)$.

19. $a(a^2-a+1)+3(a^2+a-2)-2(a^2+2a-3)$.

20. $3x(x^2-2x+2)-2x(3x^2+4x-5)+x(4x^2+5x-6)$.

21. $-2a(b-c+d)-3a(c-d+b)-a(d-b-c)$.

Solve and verify:

22. $3(x-1)=2(x+4)$.

23. $5(x-2)-2(x+2)=70$.

24. $6(2x-3)-3(x-3)=0$.

25. $2(5x-9)+4(x-11)=36$.

26. $3(x+2)+5(x-3)=2(x-4)+4(x-1)+13$.

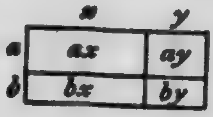
27. Find the sum of $x(x+1)$, $3x(x-2)$, $2x(x-5)$.

28. Subtract $a(2a^2-a+1)$ from $2a(a^2+3a-2)$.

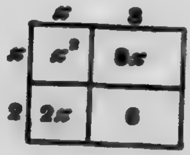
29. If a stands for x^2+xy+y^2 and b for x^2-xy+y^2, find the values of $a-b$, $2a+b$, $3a-2b$.

41. Multiplication by a Compound Quantity.

The measures of the sides of the large rectangle are $a+b$ and $x+y$. The measure of the area is the product of $a+b$ and $x+y$, which is seen to be $ax+ay+bx+by$,



$$\therefore\ (a+b)(x+y)=ax+ay+bx+by.$$



This diagram shows how to find the product of $x+3$ and $x+2$. What does it show the product to be?

Make a similar figure which will show the product of $a+b$ and $a+b$, and thus find the value of $(a+b)^2$, or the square of $a+b$.

The method of obtaining the product without the diagram is similar to that used in arithmetic.

In arithmetic.

$$\begin{array}{rl}
12= & 1.10+2 \\
23= & 2.10+3 \\
\hline
12\times\ 3=\ 36= & 3.10+6 \\
12\times 20=240= & 2.10^2+4.10 \\
\hline
12\times 23=276= & 2.10^2+7.10+6
\end{array}$$

In algebra.

$$\begin{array}{rl}
x+2 & \\
2x+3 & \\
\hline
3x+6= & 3(x+2) \\
2x^2+4x\quad = & 2x(x+2) \\
\hline
2x^2+7x+6= & (2x+3)(x+2)
\end{array}$$

Thus, *the product of any two expressions is obtained by multiplying each term of the multiplicand by each term of the multiplier. The proper signs are attached to these partial products, and the sum of the partial products is then taken.*

In multiplying in arithmetic we begin at the right, but in algebra it is usual, but not necessary, to begin at the left.

Ex.—Multiply (1) $2a-3b$ by $3a-2b$.
(2) $3x-5y$ by $4x+y$.

$$\begin{array}{rr|rr}
\text{Check} & (1) & \text{Check} & (2) \\
a=b=1 & 2a-3b & x=y=1 & 3x-5y \\
-1 & 3a-2b & -2 & 4x+y \\
1 & \overline{6a^2-9ab} & 5 & \overline{12x^2-20xy} \\
\overline{-1} & -4ab+6b^2 & \overline{-10} & +3xy-5y^2 \\
 & \overline{6a^2-13ab+6b^2} & & \overline{12x^2-17xy-5y^2}
\end{array}$$

42. Checking Results. In Chapter II. we saw how to verify the root which we obtained in solving an equation. We might verify our work in subtraction by addition. As in addition, the work in multiplication is easily checked by substituting particular numbers for the letters involved.

Thus to check the work in the first example in the preceding article, we might substitute 1 for each letter involved.

When $a=b=1$, $\quad 2a-3b=2-3=-1,$
$\qquad\qquad\qquad\quad 3a-2b=3-2=1,$
and $\qquad 6a^2-13ab+6b^2=6-13+6=-1.$

Since the product of -1 and 1 is -1, the work is *likely* correct. A convenient way of exhibiting the test is shown. Of course any numbers might be used in checking, but we naturally choose the simplest ones.

EXERCISE 30

Find the product of the following and check:

$$\begin{array}{llll}
1.\ \begin{array}{r} x+3 \\ \underline{x+4} \end{array} &
2.\ \begin{array}{r} 2x+7 \\ \underline{x+1} \end{array} &
3.\ \begin{array}{r} x+5 \\ \underline{2x+2} \end{array} &
4.\ \begin{array}{r} 3x+4 \\ \underline{2x+3} \end{array}
\end{array}$$

5. $\begin{array}{r} a-3 \\ \underline{a-4} \end{array}$ 6. $\begin{array}{r} a-5 \\ \underline{a+3} \end{array}$ 7. $\begin{array}{r} b-4 \\ \underline{2b-3} \end{array}$ 8. $\begin{array}{r} 3a-5 \\ \underline{2a+5} \end{array}$

9. $\begin{array}{r} 2x-3 \\ \underline{2x+3} \end{array}$ 10. $\begin{array}{r} x+y \\ \underline{x-y} \end{array}$ 11. $\begin{array}{r} 2x-3a \\ \underline{5x-\ \ a} \end{array}$ 12. $\begin{array}{r} 3a-7c \\ \underline{3a+7c} \end{array}$

13. $(3a+4b)(2a-5b)$. 14. $(x-5y)(2x+7y)$.

15. $(a+b)(2c-d)$ 16. $(a-3b)(3c-4d)$.

17. Find the square of $x-y$ by multiplying it by $x-y$. What is $(x-y)^2$ equal to?

18. Find the squares of $2a-b$, $2a-3b$, $4a+5$, $3a+4b$. Check by putting $a=3$, $b=1$.

19.* Simplify $(x+1)(x+2)+(x-2)(x+3)$.

20. Simplify $3(a+2)(a-2)+2(a-5)(a+1)$.

21. When $8x^2-2x-15$ is divided by $2x-3$ the quotient is $4x+5$. Prove that this is correct.

22. Show that $(6x-8)(2x-3)=(4x-6)(3x-4)$.

23. $m(x+y)=mx+my$. Find the value of $mx+my$ when $m=2{\cdot}14$, $x=43{\cdot}7$, $y=56{\cdot}3$.

24. If a train goes $2a-3b$ miles per hour, how many miles will it go in $2a+3b$ hours?

25. Simplify $(x+y)^2+(x-y)^2$; $(x+y)^2-(x-y)^2$.

26. Simplify $2(a-b)(2a+b)-3(a+b)(a-2b)$.

27. Subtract $(x+2)(x-9)$ from $(x+3)(x+4)$.

28. Multiply $3(x+3)-2(x+4)$ by $2(x-5)-(x-3)$.

29. Subtract $(x+3)(x+7)$ from $(x+1)(x+11)$. For what value of x are these quantities equal? (Verify.)

30. Show that there is no value of x which will make $(x-10)(x-1)$ equal to $(x-3)(x-8)$.

31. Subtract the sum of $(3x+2)(2x+3)$ and $(3x-2)(2x-3)$ from the sum of $(4x+3)(3x+4)$ and $(4x-3)(3x-4)$.

32. Simplify $(x-3)^2+(x-2)(x+2)+(x+1)(x+5)$.

33. Subtract the product of $2a-5$ and $3a+2$ from the product of $3a+5$ and $2a-3$.

34. Find the sum of the squares of $x+1$, $x+2$, $x+3$. Check by putting $x=2$.

Solve and verify:

35. $(x+5)(x-7)=(x-5)(x+2)$. **36.** $(x-1)^2=(x-5)(x+2)$.

37. $(2x-1)(3x-1)=(x-2)(6x+4)$.

38. $(x+11)(x-2)=(x-7)(x-1)+107$.

39. $x(x+1)+(x+1)(x+2)=2(x+1)(x+3)$.

40. $(x+1)^2+(x+2)^2+(x+3)^2=3(x-2)^2+14$.

42. Division by a Simple Positive Quantity.

To divide 24 by 6 is the same as to find the number by which 6 must be multiplied to produce 24.

284

Thus division is the inverse of multiplication as subtraction is the inverse of addition.

Since

$$a \times b = ab,$$

$$\therefore\ ab \div a = b \text{ and } ab \div b = a.$$

If we wish to divide $6xy$ by $2x$, we must find what $2x$ must be multiplied by to produce $6xy$.

(1) 2 must be multiplied by 3 to produce 6,
(2) x „ „ „ „ y „ „ xy.

$$\therefore\ 6xy \div 2x = 3y.$$

Similarly, $$15abc \div 3bc = 5a.$$

17
12
119
16
9
14
412
289

A problem in division may be written in the fractional form, the dividend being the numerator of the fraction and the divisor the denominator.

Thus, $$\frac{24}{6}=4,\quad \frac{ab}{a}=b,\quad \frac{6xy}{2x}=3y.$$

13
13
39

As in arithmetic, we may remove or cancel from the dividend and divisor any factor which is common to both

Thus, $\frac{12ab}{3b}=4a$, on removing the factors 3 and b.

Similarly, $$\frac{15axy}{5x}=3ay \text{ and } \frac{18pqr}{9qr}=2p.$$

24
3
72

-16x
-18x
-34x

13
16

44. Index Law for Division.

Since $a^2 \times a^3 = a^5$ by the index law for multiplication,

$$\therefore\ a^5 \div a^2 = a^3 \text{ or } a^5 \div a^3 = a^2,$$

or $$\frac{a^5}{a^2} = \frac{\cancel{a} . \cancel{a} . a . a . a}{\cancel{a} . \cancel{a}} = a^3 \text{ and } \frac{a^5}{a^3} = \frac{\cancel{a} . \cancel{a} . \cancel{a} . a . a}{\cancel{a} . \cancel{a} . \cancel{a}} = a^2.$$

Thus, *the index of the quotient of powers of the same quantity is found by* **subtracting** *the index of the divisor from the index of the dividend.*

Thus, $x^6 \div x^2 = x^{6-2} = x^4$; $a^7 \div a^2 = a^{7-2} = a^5$.

Similarly, $\frac{15a^3b^3}{3a^2b} = 5a^{3-2}b^{3-1} = 5ab^2$.

The work in division may be verified by multiplication. Thus the preceding division is seen to be correct, since $5ab^2 \times 3a^2b = 15a^3b^3$.

EXERCISE 31

Copy and supply the quotients, verifying the results by mental multiplication:

1. $\frac{3xy}{x}$.
2. $\frac{5abc}{ab}$.
3. $\frac{24mn}{3n}$.
4. $\frac{25xyz}{5z}$.
5. $\frac{4a^2}{2a}$.
6. $\frac{42x^3}{7x}$.
7. $\frac{12a^4}{2a^3}$.
8. $\frac{65m^2n}{13mn}$.
9. $\frac{3a^2b^2c}{abc}$.
10. $\frac{18p^3q^2}{6p^2}$.
11. $\frac{4a^2}{\frac{1}{2}a}$.
12. $\frac{12x^2y^2}{\frac{1}{2}xy}$.
13. $6xy \div 2x$.
14. $10a^3 \div 5a$.
15. $\frac{1}{2}mv^2 \div \frac{1}{2}v$.
16. $10x^5 \div 2x^3$.
17. $16a^2b \div 4ab$.
18. $15x^3y^3z^4 \div 3x^2yz^3$.
19. $22a^4b^3 \div 11a^2b^2$.

45. Rule of Signs for Division.

Since $(+a)\times(+b)=+ab$, $(-a)\times(+b)=-ab$, $(+a)\times(-b)=-ab$, $(-a)\times(-b)=+ab$, it follows that

$$\frac{+ab}{+a}=+b,\quad \frac{-ab}{-a}=+b,\quad \frac{-ab}{+a}=-b,\quad \frac{+ab}{-a}=-b.$$

When is the sign of the quotient + and when is it − ? What then is the rule of signs for division ?

Compare it with the rule of signs for multiplication (art. 38).

Ex.—Divide $-10x^2y^3$ by $-2xy^2$.

(1) What is the sign of the quotient?
(2) What is the numerical coefficient?
(3) What is the literal part?
(4) What is the complete quotient?

EXERCISE 22 (Oral)

Perform the indicated divisions:

1. $12\div-3$. -4
2. $-12\div-4$. +3
3. $-10\div2$. - 5
4. $-7\div7$.
5. $-2a\div-a$.
6. $-12\div2\div-3$.
7. $0\div5$.
8. $0\div-5$.
9. $6a^2\div-2a$.
10. $ab\div-a$.
11. $axy\div-x$.
12. $45\div-5\div-3$.
13. $10a^5\div-2a^2$.
14. $-6a^3\div3a$.
15. $27x^4\div-3x^2$.
16. $-12m^2n\div-6mn$.
17. $x^2y^2z\div-xyz$.
18. $-4a^6\div-2a^3$.
19. $\frac{-p^2qr}{-pq}$.
20. $\frac{36m^3n^3}{-4mn^2}$.
21. $\frac{6\cdot4x^3}{-2x}$.
22. Fill in the blanks in the following:

+3
5
65

	(1)	(2)	(3)	(4)	(5)
Dividend:	$6a^3$	$-10x^2$	$-10abc$	——	
Divisor:	$2a$	——	——	$-5c$	$35m^2n$
Quotient:	——	$-2x$	$2ac$	$6a$	$-5m$
					——

46. Division of a Compound Quantity by a Simple One.

If we divide 6 ft. 4 in. by 2 we get 3 ft. 2 in., or 12 lb. 6 oz. by 6 we get 2 lb. 1 oz.

Similarly,

$$3\overline{)\,9\text{ ft. }6\text{ in.}}\qquad 4\overline{)\,16\text{ lb. }8\text{ oz.}}\qquad 2\overline{)\,6\text{ tens}+8\text{ units}}$$
$$\ \ 3\text{ ft. }2\text{ in.}\qquad\quad 4\text{ lb. }2\text{ oz.}\qquad\quad 3\text{ tens}+4\text{ units}$$

$$3\overline{)\,9f+6i}\qquad 4\overline{)\,16a+8b}\qquad 2\overline{)\,6t+8}$$
$$\ \ 3f+2i\qquad\quad 4a+2b\qquad\quad 3t+4$$

$$a\overline{)\,ab+ac}\qquad 3x\overline{)\,6x^3-3x^2}\qquad -ab\overline{)\,3a^2b^2-2ab}$$
$$\ \ b+c\qquad\quad 2x^2-x\qquad\quad -3ab+2$$

Thus it is seen, that *we divide a compound expression by a simple one by dividing each term of the dividend by the divisor, attaching the proper sign to each term of the quotient.*

EXERCISE 33 (1-15, Oral)

Divide the first quantity by the second:

1. $9a^2+6a$, 3.
2. $6x^4+4x^3-2x$, x.
3. $15x^2-10x$, $5x$.
4. $16m^2-4m$, $4m$.
5. x^2y+xy^2, xy.
6. $12a^2-4ab$, $-2a$.
7. $-ax+ay$, $-a$.
8. a^4+a^2-a, a.
9. $6x^2-4xy$, $2x$.
10. $-6ab--6a$, $-3a$.
11. $6a^3-8a^2+4a$, $-2a$.
12. $a^2b^3-a^3b^2$, ab^2.
13. $-5a^4-10a^3$, $-5a^2$.
14. $-4x+10x^2-6x^3$, $-2x$.
15. $3y^3-2y^2$, $\frac{1}{2}y$.

Simplify:

16. $\dfrac{3x+6}{3}+\dfrac{10x-15}{5}$.
17. $\dfrac{ab+ac}{a}+\dfrac{bc+ab}{b}+\dfrac{ac+bc}{c}$.
18.* $\dfrac{a^2+3a}{a}-\dfrac{3a^2+6a}{3a}$
19. $\dfrac{(x+2)(x-2)+(x-2)(x-4)}{2}$.
20. $\dfrac{x^2+xy}{x}+\dfrac{y^2-xy}{y}$.
21. $\dfrac{(a+2)(a+3)-(a-3)(a-2)}{2a}$.
22. $\dfrac{ab-ac}{-a}+\dfrac{bc-ab}{-b}$.
23. $\dfrac{a^3-a^2}{a}+\dfrac{a^2-a}{a}+1$.

24. Subtract $(x+3)(x-8)$ from $(2x-4)(x+6)$ and divide the remainder by x.

25. Solve and verify $\dfrac{x^2-10x}{x}+\dfrac{3x^2+15x}{3x}+\dfrac{10x-15}{5}=40$.

EXERCISE 34 (Review of Chapter V)

1. State the rule of signs for multiplication and for division.

2.* If $a=3$ and $b=-4$, find the values of:

$$a^2,\ b^2,\ ab,\ a^2+b^2,\ a^2-b^2,\ a^3,\ b^3,\ a^3-b^3.$$

3. What are the values of $(-1)^3$, $(-1)^5$, $(-1)^{10}$, $(-2)^4$, $(-3)^3$?

4. Simplify $3a^2\times -4b^2\times -2ab\div 6ab^3$.

5. Simplify $2a(a+3)+3a(2a-5)$.

6. What is the area in square feet of a rectangle which is $(a+b)$ feet long and $(a-b)$ yards wide?

7. Make a diagram to show that $3a\times 4a=12a^2$.

8. A merchant bought a pieces of silk at 60 cents a yard and b pieces at 80 cents a yard. If each piece contained 50 yards, find the total cost in dollars.

9. To the product of $3x-2$ and $2x-3$ add the product of $3x+2$ and $2x+3$.

10. From the product of $5x-3y$ and $2x+y$ subtract the product of $3x-2y$ and $2x-3y$.

11. Make a diagram to show that the product of $a+3$ and $a+1$ is a^2+4a+3.

12. Divide $4a^5-6a^3-8a$ by $-2a$ and verify.

13. To the square of $2m-3n$ add the square of $3m-2n$.

14. Prove that when $15x^2-8xy-12y^2$ is divided by $5x-6y$ the quotient is $3x+2y$.

15. Find the product of $a-b$, $a+b$ and a^2+b^2. Check by substituting 3 for a and 2 for b.

16. Simplify $\dfrac{4x^3-8x^2+12x}{4x}+\dfrac{15x^3+10x^2-15x}{5x}$.

17. Solve $(2x+3)(3x+2)=(6x-1)(x+3)$. (Verify.)

18. Simplify $(2a-3b)(a+b)+(a-b)(3a+b)$.

19. What value of x will make $(x+3)(x+9)$ equal to $(x+5)(x+6)$? Could $(x+3)(x+9)$ be equal to $(x+4)(x+8)$?

20. Find the sum of $(a-1)^2$, $(a-2)^2$ and $(a-3)^2$.

21. Subtract the product of $2x-3y$ and $3x+2y$ from the product $3x-4y$ and $4x+3y$.

22. Simplify $\dfrac{4a^3+2a}{2a}+(3+2a)(1-a)$.

23. Find the value of $2x^2+3x-1$, when $x=-3$; when $x=-4$.

24. Find the product of $x-2$, $x+2$ and x^2+4.

25. If $x=a^2-3a+1$ and $y=2a^2-a-1$, find the values of $2x+3y$, $4x-2y$, $\dfrac{x+y}{a}$.

26. If $x=2a+b$ and $y=a+2b$, find in terms of a and b the values of $\dfrac{ax-by}{2}$, $\dfrac{4x-2y}{3a}$, $\dfrac{x^2-y^2}{3}$.

27. If $x=3b-2c$ and $y=2b-3c$, find the value of $(2x-y)(3x-2y)$.

28. If $x=2$, $y=2$, $z=-4$, find the value of $x^3+y^3+z^3-3xyz$.

CHAPTER VI

SIMPLE EQUATIONS *(continued from Chapter II.)*

47. Definition. An equation is the statement of the equality of two algebraic expressions.

Thus, $2x+3=13$ is an equation, and the solving of it consists in finding a value of x which will make the statement true.

The beginner should clearly see the difference between the value of x in an **expression** like $2x+3$ and the value of x in an **equation** like $2x+3=13$.

In the expression $2x+3$, x may represent any number, and for different values of x the expression has different values. But in the equation $2x+3=13$, x can not represent any number we please, but some particular number, in this case 5, which when substituted for x will make $2x+3$ have the value 13.

48. Identity. The statement $4(x-2)=4x-8$ is an equation according to the definition we have given.

If the first side of this equation be simplified by multiplication, we obtain $4x-8$, which is *identically* the same as the second side. It is at once seen that this equation is true for all values of x.

An equation which is true for all values of the letters involved is called an identical equation or briefly an identity, while an equation which is true only for certain values of the letters involved is called a conditional equation. The usual method, however, is to call all conditional equations simply "equations," and all identical equations, "identities."

Thus, $5x-2=3x+10$, is an equation,

and $(x+3)(x-3)=x^2-9$, is an identity.

We cannot always see mentally whether a given statement is an equation or an identity.

Thus, $(x+2)(x+3)=(x-1)(x-3)+3(3x+1)$ might appear to be an equation, but if we simplify each side, we find that each becomes x^2+5x+6, and this statement is therefore an identity.

EXERCISE 35

Which of the following statements are equations and which are identities?

1. $8(x+3)=4x+4(x+6)$.
2. $3x(x+7)=x(x+1)+2x(x+5)+10$.
3. $(x-3)^2-5=x(x-6)+4$.
4. $(2x-4)(x-5)+(x-2)(x-3)=(3x-2)(x-7)+40$.
5. $(x+a)(x^2+a^2)=x^3+ax(x+a)+a^3$.
6. $(x+2)(x-3)=x(x+5)+3(x-1)$.

40. Transposition of Terms. In Chapter II. the method of solving easy equations was dealt with.

The method depended almost entirely on the proper use of the four axioms of art. 15.

The following examples will show how the methods of Chapter II. may be abbreviated.

Ex. 1.—Solve $7x-6=4x+12$.

Add 6 to each side, $7x=4x+12+6$.

Subtract $4x$ from each, $7x-4x=12+6$.

Collect terms on each side, $3x=18$.

Divide each side by 3, $x=6$.

Here we added 6 to each side with the object of causing the -6 to disappear from the first side of the equation, so that we might have only unknown quantities on that side. But the addition of 6 to the second side caused $+6$ to appear on that side.

We might say then, that the -6 was transposed from the first side and written on the other side with its *sign* *changed*, and similarly, that the $4x$ was transposed from the second side to the first, with its sign changed.

We therefore have the following rule :

Any quantity may be transposed from one side of an equation to the other if the sign of the quantity be changed.

Using the rule, the solution of Ex. 1 might appear thus :

$$7x-6=4x+12.$$

Transposing terms, $7x-4x=12+6,$

$$\therefore\ 3x=18,$$

$$\therefore\ x=6.$$

Verify this result.

Ex. 2.—Solve $2(3x-5)+3(x-5)=7(x-1)$.

Removing brackets, $6x-10+3x-15=7x-7.$

Transposing terms, $6x+3x-7x=10+15-7$

$$\therefore\ 2x=18,$$

$$\therefore\ x=9.$$

Verification, when $x=9$:

first side $=2\times22+3\times4=56,$

second side $=7\times8\qquad=56.$

Ex. 3.—Solve $3(y-2)-5(y-3)=17$.

Removing brackets, $3y-6-5y+15=17.$

Transposing terms, $3y-5y=6-15+17,$

$$\therefore\ -2y=8,$$

$$\therefore\ y=\frac{8}{-2}=-4.$$

Verification : first side $=3(-6)-5(-7)$

$$=-18+35=17.$$

Ex. 4.—Solve $(2x-1)^2-(x-3)(x-2)=3(x-2)^2-4$.

Here the indicated multiplications are first performed.

$$(2x-1)^2=4x^2-4x+1,$$

$$(x-3)(x-2)=x^2-5x+6,$$

$$(x-2)^2=x^2-4x+4,$$

$$\therefore\ 4x^2-4x+1-(x^2-5x+6)=3(x^2-4x+4)-4,$$

$$\therefore\ 4x^2-4x+1-x^2+5x-6=3x^2-12x+12-4,$$

$$\therefore\ 4x^2-x^2-3x^2-4x+5x+12x=12-4-1+6,$$

$$\therefore\ 13x=13,$$

$$\therefore\ x=1.$$

Here the product of $x-3$ and $x-2$ is first found and enclosed in brackets with the minus sign preceding. In the next line the brackets are removed and the signs changed.

In $3(x-2)^2$, the $x-2$ must first be squared and the product multiplied by 3.

NOTE.—The beginner should not attempt to perform these double operations together.

EXERCISE 35

Solve and verify :

1. $4x-4=2x+8$.
2. $3x-7=8-2x$.
3. $3-3x=9-5x$.
4. $2(x-5)=x+20$.
5. $5(y-2)=3(y+4)$.
6. $10(x-3)=8(x-2)$.
7. $11(4x-5)=7(6x-5)$.
8. $7x-11+4x-7=3x-8$.
9. $14+5x=9x-11+3$.
10. $3(5x-6)-9x=30$.
11. $7(x-3)=9(x+1)-38$.
12. $5(x-7)+63=9x$.
13. $72(x-5)=63(5-x)$.
14. $28(x+9)=27(46-x)$.
15. $7(4x-5)=8(3x-5)+9$.
16. $4(x+2)=3-3(2x-5)$.
17. $(x+7)(x-3)=(x-1)(x+1)$.
18. $(x-8)(x+12)=(x+1)(x-6)$.
19. $20(x-4)-12(x-5)=x-6$.
20. $5(2x-1)-3(4x-6)=7$.
21. $(2m-5)(4m-7)=8m^2+52$.
22. $5(3h+1)-7h-3(h-7)=6$.
23. $(x+5)^2-(x+3)^2=40$.
24. $(x+5)^2-(4-x)^2=21x$.
25. $4(2y-7)-3(4y-8)=$ [illegible] $y-7$.
26. $(x+4)(x-3)-(x+2)(x+1)=42$.
27. $(2x-7)(x+5)=(2x-9)(x-4)+229$.
28. $(x+1)^2+(x+2)^2+(x+3)^2=3(x+1)(x+4)-7$.
29. $2(x-1)^2-3(x-2)(x+3)=32-(x-3)(x-4)$.
30. What value of x will make $10x+11$ equal to $5x-9$?
31. Prove that $3(x-2)+4(3x-5)=5(3x-6)+4$ is true for all values of x.
32. What value of a will make $5(a-3)$ exceed $3(a-7)$ by 28 ?
33. For what value of x will the sum of $12+7x$, $4x+3$ and $9-5x$ be zero ?

34. If $x=2$ is a solution of the equation

$$(x+1)(x+2)=(x-4)(x-5)+k,$$

find the value of k.

35. Prove that 10 is a root of the equation

$$(x+3)(x+4)+(x+5)(x+6)=422.$$

36. When $(3x+2)(4x-5)$ is subtracted from $(2x+7)(6x+3)$ the remainder is 141. Find x.

37. What value of y will make $(y-3)(y+3)$ exceed $(y+4)(y-7)$ by 40 ?

38. What value of k will make $(5-3k)(7-2k)$ equal to

$$(11-6k)(3-k) \text{ ?}$$

39. What is peculiar about the equation

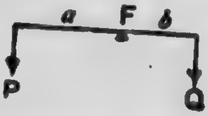
$$(x-5)^2-(x-3)(x-7)=0 \text{ ?}$$

40. Under what condition is the square of $x+3$ equal to the product of $x-1$ and $x+6$?

41. If $3(2x-1)$ is greater than $12(x-3)$ by the same amount that $5x$ is greater than 22, find x.

42. If $4ax-4a^2=ax+2a^2$, what is the value of x ?

43. The lever in the diagram is balanced by the weights P and Q when $Pa=Qb$. The point of support F is called the fulcrum. If $P=10$ lb., $Q=15$ lb. and $a=12$ in., what is the length of b ?



44. Two boys balance on a teeter 16 feet in length. The heavier boy weighs 85 lb. and the point of support is 6 feet from his end of the teeter. Find the weight of the other boy.

45. How far from the larger weight must the fulcrum be placed, if weights of 8 lb. and 16 lb. balance at opposite ends of a lever 12 feet long ?

46. The formula $C=\frac{5}{9}(F-32)$ is used to change Fahrenheit readings of a thermometer to Centigrade readings. If $F=77°$, find the value of C.

47. Change the following readings to Fahrenheit readings :

$$0°\text{C.}, 40°\text{C.}, 100°\text{C.}, -10°\text{C.}, -50°\text{C.}$$

48. What is the temperature when the two scales indicate equal numbers ?

50. Equations with Fractional Coefficients.

Ex. 1.—Solve $\frac{1}{2}x+\frac{1}{3}x=20$.

Since $\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$,

$$\therefore\ \tfrac{5}{6}x=20,$$
$$\therefore\ x=20\div\tfrac{5}{6}=24.$$

Instead of adding the fractions, we might get rid of them by multiplying each term of the equation by 6.

Then

$$\tfrac{1}{2}x\times 6+\tfrac{1}{3}x\times 6=20\times 6,$$
$$\therefore\ 3x+2x=120,$$
$$\therefore\ 5x=120,$$
$$\therefore\ x=24.$$

Verify by substituting in the *original* equation.

Ex. 2.—Solve $\frac{1}{2}(x+1)+\frac{1}{3}(x+2)=\frac{1}{4}(x+14)$.

Multiply each quantity by 12 (the L.C.M. of 2, 3, 4),

$$\therefore\ \tfrac{1}{2}(x+1)\times 12+\tfrac{1}{3}(x+2)\times 12=\tfrac{1}{4}(x+14)\times 12,$$
$$\therefore\ 6(x+1)+4(x+2)=3(x+14).$$

Complete the solution and verify.

Ex. 3.—Solve $\dfrac{x-2}{5}-\dfrac{x-3}{6}=\dfrac{x-7}{10}$.

Multiply by 30, $\therefore\ \dfrac{x-2}{5}\times 30-\dfrac{x-3}{6}\times 30=\dfrac{x-7}{10}\times 30$,

$$\therefore\ 6(x-2)-5(x-3)=3(x-7),$$
$$\therefore\ 6x-12-5x+15=3x-21,$$
$$\therefore\ 6x-5x-3x=12-15-21,$$
$$\therefore\ -2x=-24,$$
$$\therefore\ x=12.$$

Verification : first side $=\frac{10}{5}-\frac{9}{6}=2-1\frac{1}{2}=\frac{1}{2}$.
second side $=\frac{5}{10}\quad=\frac{1}{2}$.

91
−36
55

NOTE.—In this solution the beginner is advised not to attempt to omit the line with the brackets. He may, however, omit the preceding line when he feels that he can safely do so.

51. Steps in the Solution of an Equation. In solving an equation the steps in the work are :

(1) *Clear the equation of fractions by multiplying each term by the L.C.M. of the denominators of the fractions.*

(2) *Remove any brackets which appear.*

96
38
58
9 3

(3) *Transpose all the unknown quantities to one side and the known quantities to the other.*

(4) *Simplify each side by collecting like terms.*

(5) *Divide each side by the coefficient of the unknown.*

(6) *Verify the result by substituting the root obtained in the original equation.*

EXERCISE 37

Solve and verify:

1. [illegible]$x=x+5$.
2. [illegible]$x=$[illegible]$x+2$.
3. [illegible]$x-$[illegible]$x=10$.
4. [illegible]$x+$[illegible]$x+$[illegible]$x=26$.
5. [illegible]$x+$[illegible]$x=x+5$.
6. $\frac{1}{2}x=\frac{2x}{3}-4$.
7. [illegible]$y=$[illegible]$y+$[illegible].
8. $\frac{x}{8}+\frac{x}{4}+\frac{x}{2}=x-4$.
9. $\frac{x}{2}-\frac{x}{5}=\frac{x}{4}+1$.
10. $\frac{3m}{2}-\frac{7m}{5}=4$.
11. $\frac{x}{2}-\frac{x}{3}=\frac{4x}{9}-15$.
12. [illegible]$x+$[illegible]$x=1$[illegible]$-x$.
13. $\frac{x}{5}+2=1$[illegible]$+\frac{x}{20}-\frac{x}{5}$.
14. [illegible]$x-$[illegible]$+7x=3x+1$[illegible].
15. $\frac{x}{3}-\frac{x}{4}=2$[illegible].
16. [illegible]$(x-3)=20$.
17. $\frac{7x+2}{5}=\frac{4x-1}{2}$.
18. $\frac{x+1}{10}-3=0$.
19. $\frac{x}{3}+\frac{x-8}{4}=5$.
20. $\frac{x-1}{4}+\frac{x+3}{5}=8$.
21. [illegible]$(x-3)+$[illegible]$(x-5)=0$.
22. [illegible]$(x-6)=$[illegible]$(x+5)+$[illegible]$(x-13)$.
23. $\frac{3x-1}{3}+\frac{5}{12}=\frac{x}{4}+\frac{2x+1}{5}$.
24. $\frac{x+2}{3}+2=\frac{x+4}{5}+\frac{x+6}{7}$.
25. $\frac{x-3}{4}=\frac{2x-4}{5}+\frac{3x-5}{8}$.
26. [illegible]$(y-3)-$[illegible]$(y-5)=1$.

27. $\frac{x+1}{4}-\frac{x-1}{6}=1.$

28. $\frac{x-7}{8}-\frac{x-3}{11}=0.$

29. $\frac{x-2}{2}-\frac{x+2}{4}=\frac{x-3}{3}.$

30. $\frac{x+1}{2}-\frac{1}{4}=x-\frac{2x-1}{3}.$

31. $\frac{x}{4}-\frac{5x+9}{6}=\frac{2x-9}{6}.$

32. $6-\frac{x-1}{2}-\frac{x-2}{3}=\frac{3-x}{4}.$

33. $5(x-2)=3{\cdot}65.$

34. $2{\cdot}34=4(x+1{\cdot}5).$

35. ${\cdot}5x-3={\cdot}25x+{\cdot}2x.$

36. ${\cdot}2(x-1)+{\cdot}5(x-9)=3.$

37. $\frac{3x-9}{7}-\frac{x+1}{11}=\frac{3x-14}{8}.$

38. $\frac{x+6}{4}-\frac{3x-16}{12}-1=\frac{x+3}{6}.$

13½ mins.

done in proof.

39. $\frac{2-x}{3}+\frac{3-x}{4}+\frac{4-x}{5}+\frac{5-x}{6}+\frac{3}{4}=0.$

40. $\frac{x-1}{9}-\frac{2-x}{4}-\frac{2x-1}{14}+\frac{2-3x}{30}=0.$

52. Problems leading to Simple Equations. In Chapter II. we saw how certain arithmetical problems might be solved by means of equations. The steps in the solution of such problems are stated in art. 19, to which the pupil should now refer.

The beginner will find his chief difficulty with step 4, in which he is required to translate the statements given in ordinary language into algebraic language.

Some examples are now given to illustrate how this translation is effected.

Ex. 1.—Find three consecutive numbers whose sum is 63.

If we let x represent the smallest one, what would represent the others? How would you now express that the sum is 63?

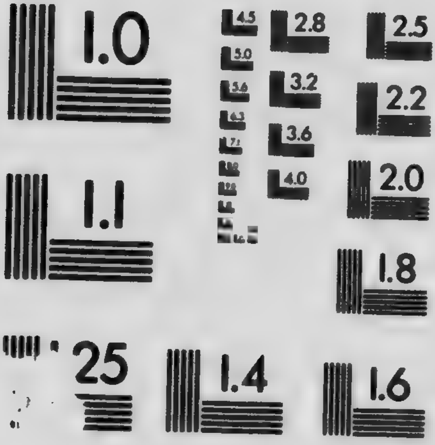
We thus obtain the equation:

$$x+(x+1)+(x+2)=63.$$

Write out the full solution of this example and verify the result.

MICROCOPY RESOLUTION TEST CHART

(ANSI and ISC TEST CHART No. 2)



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Ex. 2.—A is 3 times as old as B; 2 years ago A was 5 times as old as B was 4 years ago. Find their ages.

Let x years represent B's age.

What will now represent A's age?

What will represent A's age, 2 years ago?

What will represent B's age, 4 years ago?

Now express that $3x-2$ is 5 times $x-4$.

The complete solution might appear thus:

Let x years $=B$'s age,

$\therefore$ $3x$ „ $=A$'s age,

$\therefore$ $(3x-2)$ „ $=A$'s age, 2 years ago,

$\therefore$ $(x-4)$ „ $=B$'s age, 4 years ago,

$$\therefore\ 3x-2=5(x-4),$$
$$\therefore\ 3x-2=5x-20,$$
$$\therefore\ 18=2x,$$
$$\therefore\ x=9.$$

$\therefore$ B's age is 9 years and A's is 27 years.

Ex. 3.—How do you represent 3% of 130? 4% of \$27? 5% of \$$x$? $2\frac{1}{2}$% of \$$(x+50)$?

Solve the problem: "Divide \$620 into two parts so that 5% of the first part together with 6% of the other part will make \$34."

Let \$$x$ = the first part,

$\therefore$ \$$(620-x)$ = the other part,

$\therefore$ $\frac{5}{100}$ of \$$x$ = 5% of the first part,

$\therefore$ $\frac{6}{100}$ of \$$(620-x)$ = 6% of the other part,

$$\therefore\ \tfrac{5}{100}x+\tfrac{6}{100}(620-x)=34,$$
$$\therefore\ 5x+6(620-x)=3400.$$

Complete the solution and verify the result.

Ex. 4.—What is the excess of 73 over 50? What is the defect of 30 from 50? What is the excess of x over 50? The defect of x from 89?

Solve the problem: "The excess of a number over 50 is 11 greater than its defect from 89. Find the number."

Let $x = \text{the number},$
then $x - 50 = \text{its excess over } 50,$
and $89 - x = \text{its defect from } 89,$
$\therefore\ x - 50 = 89 - x + 11.$

Complete the solution and verify.

Ex. 5.—The value of 73 coins consisting of 10c. pieces and 5c. pieces is \$5. How many are there of each ?

Let $x = \text{the number of 10c. pieces},$
$\therefore\ 73 - x =$,, ,, ,, 5c. ,,

The value of the 10c. pieces $= 10x$ cents.
The value of the 5c. pieces $= 5(73 - x)$ cents,
$\therefore\ 10x + 5(73 - x) = 500.$

Complete the solution and verify.

The pupil should be careful to express each term of the equation in the **same denomination.**

Why would it be incorrect to say that

$$10x + 5(73 - x) = 5\ ?$$

EXERCISE 38

All results should be verified.

1. A number is multiplied by 23 and 117 is then added. The result is 232. Find the number.

2. From the double of a number 7 is taken. The remainder is 95. Find the number.

3. Three times a number is subtracted from 235 and the result is 217. Find the number.

4. Five times a number with 33 added is equal to 7 times the number with 18 added. Find the number.

5. Find a number such that the sum of its third and fourth parts may be 35.

6. *A* has \$10 more than 3 times as much as *B*, and they together have \$250. How much has each ?

7. The sum of two numbers is 81. The greater exceeds 6 times the less by 4. Find the numbers.

8. Find a number whose seventh part exceeds its eighth part by 2.

9. The excess of a number over 42 is the same as its defect from 59. Find the number.

10. Find 3 consecutive numbers whose sum is 129.

11. Divide 114 into three parts so that the first exceeds the second by 15 and the third exceeds the first by 21.

12. Divide \$176 among A, B and C so that B may have \$16 less than A and \$8 more than C.

13. A man sold a lot for \$2280 and gained 14% of the cost. What did the lot cost ?

14. Divide 420 into 3 parts so that the second is double the first and the third is the sum of the other two.

15. A man buys 8 horses at \$$x$ each, 5 at \$$(x+5)$ each and 3 at \$$(x+25)$ each. The total cost is \$2020. Find x.

16. Find a number which exceeds 31 by the same amount that $\frac{2}{3}$ of the number exceeds 1.

17. Find a number which when multiplied by 6 exceeds 35 by as much as 35 exceeds the number.

18. A farmer sells 7 cows and 17 pigs for \$754. Each cow sells for \$70 more than each pig. What is the price of each cow ?

19. If 10 be subtracted from a number, 40 more than $\frac{1}{3}$ the remainder is 30 less than the number. Find the number.

20. Find two consecutive numbers such that the sum of $\frac{1}{3}$ of the less and $\frac{1}{4}$ of the greater is 44.

21. Divide 46 into two parts so that if the greater part is divided by 7 and the other by 3, the sum of the quotients is 10.

22. Divide 237 into two parts so that one part may be contained in the other $1\frac{1}{2}$ times.

23. A horse was sold for \$116·25 at a loss of 7%. What did he cost ?

24. The difference between the squares of two consecutive numbers is 17. Find the numbers.

25. A box contains two equal sums of money, one in half-dollars and the other in quarters. If the number of coins is 30, how much money is in the box ?

26. A is 35 years old ; B is 7 years old. In how many years will A be twice as old as B ?

27. My age in 20 years will be double what it was 10 years ago. What is my age ?

28. *A* is 35, *B* is 7 and *C* is 5 years old. How long will it be before *A*'s age is the sum of the ages of *B* and *C* ?

29. Find three consecutive even numbers such that the sum of a fourth of the first, a half of the second and a fifth of the third is 17.

30. *A*'s share of \$705 is $\frac{4}{5}$ of *B*'s and *B*'s is $\frac{3}{4}$ of *C*'s. What is the share of each ?

31. The simple interest on a sum at 2% together with the interest on a sum twice as large at $3\frac{1}{2}$% is \$135 per annum. What are the sums ?

32. Three % of a certain sum together with 4% of a sum which is \$50 greater is \$12·50. Find the sums.

33. The value of 52 coins made up of quarters and ten-cent pieces is \$10. How many are there of each ?

34. A square floor has a margin 2 feet wide all around a square carpet. The area of the margin is 160 sq. ft. Find the dimensions of the room.

35. In any triangle the sum of the angles is 180°. The greatest angle is 35° larger than the smallest angle and 10° larger than the other angle. Find the angles.

36. The length of a room exceeds the width by 4 feet. If each dimension be increased by 2 feet the area will be increased by 52 sq. ft. Find the length.

37. If I walk m miles at 4 miles per hour and $m+2$ miles at 3 miles per hour, the whole journey will take 15 minutes longer than if I walked at the uniform rate of $3\frac{1}{2}$ miles per hour. Find the length of the journey.

38. *A* and *B* together have \$65, *B* and *C* have \$100, *C* and *A* have \$95. How much has each ?

39 State problems which will give rise to the following equations:

(1) $5x-10=60$.

(2) $4x-x=24$.

(3) $\frac{x}{2}+\frac{x}{3}=x-10$.

(4) $23-5x=4x-4$.

40. A fruit dealer buys apples at the rate of 5 for 3 cents and sells them at the rate of 3 for 5 cents. How many must he sell to gain \$1·28 ?

A

41. The sum of two numbers is 147 and $\frac{1}{3}$ of the less is 9 greater than $\frac{1}{5}$ of the other. Find the numbers.

42. John has $\frac{3}{4}$ as much money as his brother, but when each has spent 25 cents, John has only $\frac{2}{3}$ as much as his brother. How much has each ?

53. Algebraic Statements of Arithmetical Theorems. If we take any two numbers, say 23 and 13, and add together their sum and their difference, we will find the result is twice the larger number.

Thus, $23+13=36$ and $23-13=10$,
and $36+10=46$, which is twice 23.

We see that it is true for the numbers 23 and 13, and we would find it true for other pairs of numbers, but we are not sure it is true for all pairs of numbers.

By the use of algebraic symbols and methods, we may show that the statement is true for every two numbers.

Let the larger number be a and the smaller b.
Their sum is $a+b$ and their difference is $a-b$.
But $(a+b)+(a-b)=a+b+a-b=2a$,
and $2a$ is twice the larger number.

Thus the statement $(a+b)+(a-b)=2a$ represents in a brief form the theorem stated at the beginning of this article. Besides stating it in a concise form it shows that it is true generally.

EXERCISE 39

Show that the following statements are true for all numbers :

1. The sum of two numbers is equal to their difference increased by twice the smaller number.

2. The difference between the sum of two numbers and the difference of the same two numbers is twice the smaller number.

3. Half of the sum of two numbers increased by half of their difference is equal to the larger number.

4. The sum of two numbers multiplied by one of them is equal to the square of that one, plus their product.

✓

5. The square of the sum of two numbers is equal to the square of their difference increased by four times their product.

6. The sum of three consecutive numbers is equal to three times the middle one.

7. If two integers differ by 2, twice the square of the integer between them is less by 2 than the sum of the squares of the two integers.

8. Read the statement $(a+b)^2+(a-b)^2=2(a^2+b^2)$ without using symbols and prove that it is true.

EXERCISE 40 (Review of Chapter VI)

1. What is an equation ? An identity ? ??????

2. What rule is followed in transposing terms ?

3. Solve and verify : $6x(2x+3)=(3x+2)(4x+3)$.

4. Is $\frac{4x-3}{2}=\frac{8x-6}{4}$ an equation or an identity ?

5. What value of x will make $5(x-3)-4(x-2)$ equal to zero ?

6. Solve $\frac{x-7}{5}+\frac{x-10}{7}=\frac{x-11}{6}+2$.

7. The sum of two numbers is 50. If 5 times the less exceeds 3 times the greater by 10, what are the numbers ?

8. Show that $x-1+\frac{x+3}{3}=\frac{5x+6}{6}+\frac{x-2}{2}$ is true for all values of x.

9. What value of x will make the product of $5-3x$ and $7-2x$ equal to the product of $11-6x$ and $3-x$?

10. If $\frac{2x-3}{2{\cdot}5}=\frac{3x-4}{12{\cdot}5}+{\cdot}262$, find x correct to two decimal places.

11. A and B invested equal sums. A gained \$200 and B gained \$2600. If B then had 3 times as much as A, how much did each invest ?

12. From a cask which is $\frac{7}{8}$ths full, 36 gallons are drawn and it is then half full. How much will the cask hold ?

13. Show that $x=6$ is a root of

$$(x-1)(x-2)(x-3)=2x(x-5)(2x-7).$$

14. A man has \$115 in \$2 bills and \$5 bills. If he has 35 bills altogether, how many of each has he ?

15. If $\frac{3x-20a}{5}+\frac{5x-6a}{3}=31$ and $a=\frac{1}{2}$, find x.

16. In a stairway there are 45 steps of equal heights. If they had been one inch higher, there would have been only 40 steps. How high is each step ?

2. – 17. Solve $\frac{x-4}{5}-\frac{x-5}{6}=\frac{x-2}{24}$.

18. Divide 150 into two parts such that if the smaller be divided by 23 and the other by 27 the sum of the quotients will be 6.

1.

19. The difference between the squares of two consecutive numbers is 51. Find the numbers.

20. A father is 30 years older than his son; five years ago he was four times as old. Find the son's present age.

21. If the sum of the fractions $\frac{2x+3}{3}$ and $\frac{x+5}{7}$ is 9, what is the numerical value of each fraction ?

22. Show that the difference between the squares of any two consecutive numbers is equal to the sum of the numbers. Show also that the sum of their squares is one more than twice their product.

3. – 23. Solve $2-(x-4+3x-5)=10-x$.

24. If the product of $x+2$ and $2x+5$ is greater than the product of $2x+1$ and $x+3$ by 127, find x.

25. Solve $\frac{1}{2}(2-3x)-\frac{3}{4}(x-4)=\frac{1}{4}-(x-5)$.

26. Divide ·75 into two parts so that three times the greater exceeds six times the less by ·75.

4. – 27. Solve $\frac{x-3}{5}+\frac{2+x}{3}-\frac{1-2x}{15}=0$.

2. 28. A man walked a certain distance at 3 miles per hour and returned by train at 33 miles per hour. His whole time was 4 hours. How far did he walk ?

29. Prove the accuracy of the following statement: "Take any number, double it, add 12, halve the result, subtract the original number, and 6 will remain."

30. Solve $\frac{2x}{3}+\frac{x+1}{4}-\frac{x-1}{2}=x-8$.

31. How many minutes is it to 10 o'clock if three-quarters of an hour ago it was twice as many minutes past 8 ?

32. What value of a will make $2(6x+a)-3(2x+a)=4(1\frac{1}{2}x-6)$ an identity ?

5. 33. Solve $(6x-2)(2x-1)-(4x-2)(3x-2)=4$.

34. A rectangular grass-plot has its length 5 yards longer than its width. A second plot, of equal area, is 5 yards longer and 3 yards narrower than the first. Find the dimensions of the first.

35. Solve $(x+1)(x+2)+(x+3)(x+4)=2x(x+12)$.

36. A man leaves his property amounting to $7500 to be divided among his wife, two sons and three daughters. A son is to have twice as much as a daughter, and the wife $500 more than all the children together. Find the share of each.

6. 37. Solve $\frac{x-2}{3}+\frac{4x+5}{6}-\frac{7x-8}{9}=0$.

38. Find an integer whose square is less than the square of the next higher integer by 37.

39. If $\frac{2x+1}{3}$ exceeds $\frac{3x-2}{4}$ by $\frac{x-2}{6}$, find x.

40. How far can I walk at 3 miles per hour and return on a bicycle at 10 miles per hour and be absent 6 hours 4 minutes ?

41. A man invested $\frac{1}{3}$ of his money at 3%, $\frac{1}{4}$ at 4%, $\frac{1}{5}$ at 5% and the remainder at 6%. If he receives an annual income of $516, how much did he invest ?

42. Prove that the product obtained by multiplying the sum of any two numbers by their difference is equal to the difference of their squares.

THE END!

CHAPTER VII

SIMULTANEOUS EQUATIONS

54. Equations with two Unknowns.

The sum of two numbers is 10. What are the numbers?

It is evident that there are many different answers to this problem. The numbers might be 1 and 9, 2 and 8, 3 and 7, etc., or $\frac{1}{2}$ and $9\frac{1}{2}$, -3 and 13, etc.

If we are also given that the difference of the numbers is 4, then only one of these answers will satisfy this new condition. The numbers would evidently be 7 and 3.

If we follow the method previously adopted and represent the required numbers by x and y, where x is the greater, the first condition would be expressed by the equation

$$x+y=10.$$

As stated, any number of pairs of values of x and y will satisfy this equation.

If the second condition be expressed in terms of the same unknowns, we have another equation

$$x-y=4.$$

It is now required to find a pair of values of x and y which will satisfy

$$x+y=10,$$

and

$$x-y=4.$$

If we add the corresponding sides of the equations we get:

$$2x=14, \quad \therefore \; x=7 \text{ and } \therefore \; y=3,$$

$$\therefore \; 7 \text{ and } 3 \text{ are the required numbers.}$$

55. Simultaneous Equations. Any equations which are satisfied by the same values of the unknowns are called **simultaneous equations.**

Thus, $x=7$, $y=3$ satisfy both of the equations

$$x+y=10 \text{ and } x-y=4.$$

To find a definite pair of values of x and y it is seen that **we must have two equations** containing these letters. To solve any problem where two numbers are to be found **we must have two conditions given,** from which the required equations may be obtained.

Ex. 1.—If 5 men and 4 boys earn \$43 in a day, and 3 men and 4 boys earn \$29 in a day, what sum does each earn in a day ?

Why do the first set of workers earn more than the second ? How much more do they earn ? How much then does one man earn ? How can we now find how much a boy earns ?

We might solve this problem algebraically, thus :

Let $\$x=$ the wages of a man for a day,
and $\$y=$ the wages of a boy for a day.

The conditions of the problem would now be expressed algebraically by the equations :

$$\$5x+\$4y=\$43,$$
$$\$3x+\$4y=\$29.$$

Or, omitting the \$ sign and using only the numbers,

$$5x+4y=43,$$
$$3x+4y=29.$$

Subtract the terms of the second equation from the corresponding terms of the first,

$$\therefore\ 2x=14,$$
$$\therefore\ x=7.$$

Substitute $x=7$ in the first equation and

$$35+4y=43,$$
$$\therefore\ 4y=8,$$
$$\therefore\ y=2,$$

$\therefore$ the roots of the equations are $x=7$, $y=2$,
$\therefore$ a man earns \$7 and a boy \$2 per day.

Verify by showing that these results satisfy the conditions of the given problem.

Ex. 2.—For 3 lb. of tea and 2 lb. of sugar I pay \$1·30, and for 5 lb. of tea and 4 lb. of sugar I pay \$2·20. What is the price of one pound of each?

How does this problem differ from the preceding?

What change might we make in the first statement so that the number of pounds of sugar would be the same as in the second statement?

Let x cents = the price of a lb. of tea,
and y cents = the price of a lb. of sugar.

Then
$$3x+2y=130, \qquad (1)$$
and
$$5x+4y=220. \qquad (2)$$

Multiply the first equation by 2 and we get

$$6x+4y=260, \qquad (3)$$
$$5x+4y=220. \qquad (2)$$

Now solve (2) and (3) as in the preceding example and verify the results you get.

Ex. 3.—Solve
$$3x+4y=39, \qquad (1)$$
$$4x+3y=38. \qquad (2)$$

Multiply (1) by 4 and (2) by 3 and we get

$$12x+16y=156,$$
$$12x+9y=114.$$

Complete the solution and verify.

Ex. 4.—Solve
$$5x-2y=44, \qquad (1)$$
$$3x+4y=42. \qquad (2)$$

Multiply (1) by 2, $\quad 10x-4y=88. \qquad (3)$

To get rid of the term containing y, we must now add instead of subtract. When we do so

$$13x=130,$$
$$\therefore\ x=10.$$

Substitute $x=10$ in (1) and $\quad y=3.$

56. Elimination. In all of the preceding examples the object has been to get rid of one of the unknowns, so that we might have an equation with only one unknown. The process by which this is done is called elimination.

Thus in Ex. 4 we eliminated the y. We might have eliminated the x equally well.

Solve Ex. 4 by first eliminating the x.

After performing the necessary multiplications, when do we *add* and when do we *subtract* to eliminate the unknown ?

EXERCISE 41

Solve for x and y and verify 1-21 :

1. $x+2y=8$, $x+y=5$.
2. $3x+5y=13$, $3x+2y=7$.
3. $6x+5y=23$, $3x+2y=11$.
4. $2x+3y=25$, $2x-3y=7$.
5. $5x-2y=18$, $2x-y=7$.
6. $5x+2y=24$, $2x+3y=14$.
7. $3x+5y=18$, $2x+3y=12$.
8. $5x-6y=31$, $6x-3y=33$.
9. $3x-2y=24$, $2x-3y=11$
10. $x+y=4$, $x-y=3$.
11. $3x+4y=5$, $6x+12y=13$.
12. $3x+2y=24$, $-2x+3y=10$.
13. $3x-4y=16$, $7x+3y=62$.
14. $2x+5y=0$, $3x-4y=23$.
15. $2y-3x=-22$, $2x+3y=32$.
16. $3x=2y+7$, $2x=3y-12$.
17. $x=3y+20$, $y=2x-20$.
18. $3x=2y$, $2x-5y=-33$.
19. $2x+13y=275$, $14x-17y=1385$.
20. $2x+3y=5x-y=17$.
21. $4x-5y=10y-14x=-10$.

22.* If $5x-y=8$ and $5y-x=20$, find the values of $x+y$ and $x-y$.

23. If $2x-5y-31=6y-9x+57=0$, find the value of $19x+13y$.

24. Solve $x+3=4-2y$, $7(x-1)+11y=6$.

25. If $ax+by$ equals 39 when a is 3 and b is 4, and equals 13 when a is 5 and b is -2, find x and y.

26. What values of x and y will make $16x-y$ and $4x+2y$ each equal to 6 ?

27. Solve $2(x-y)+3(x+y)=$ [illegible], $3(2x-y)+5(x-2y)=53$.

87. Fractional Equations in two Unknowns. If the equations contain fractional coefficients of x or y, the fractions may be removed by multiplication.

Ex.— Solve

$$\tfrac{1}{2}x+\tfrac{1}{3}y=8, \qquad (1)$$

$$\tfrac{1}{4}x+\tfrac{5}{2}y=32. \qquad (2)$$

Multiply (1) by 6, $3x+2y=48$.
Multiply (2) by 4, $x+10y=128$.

Complete the solution and verify.

EXERCISE 42

Solve and verify 1-20:

1. $\tfrac{1}{2}x+\tfrac{1}{3}y=3$, $x+y=7$.

2. $\tfrac{1}{3}x+y=6$, $x+\frac{y}{2}=14$.

3. $\tfrac{1}{2}(x+y)=9$, $\tfrac{1}{3}(x-y)=4$.

4. $\frac{x}{6}+\frac{y}{5}=14$, $\frac{x}{9}+\frac{y}{2}=24$.

5. $\tfrac{1}{2}x+3y=2$, $x+4y=0$.

6. $\frac{x}{8}+\frac{y}{3}=15$, $\frac{x}{4}-\frac{y}{5}=4$

7. $\tfrac{3}{4}x-\tfrac{1}{3}y=1$, $\tfrac{2}{3}x+\tfrac{3}{4}y=26$.

8. $\frac{x}{3}+\frac{y}{8}=41$, $3x-4y=0$.

9. $\frac{5x}{9}+9y=91$, $9x+\frac{5y}{9}=167$.

10. $\frac{x}{16}+\frac{y}{24}=1$, $\frac{x}{4}-\frac{y}{12}=1$.

11. $x=\tfrac{2}{3}y$, $9y-11x=80$.

12. $\tfrac{1}{3}x+\tfrac{1}{4}y=6$, $y-\tfrac{1}{4}(x-y)=7$.

13. $\cdot 3x+\cdot 5y=\cdot 23$, $6x+5y=2\cdot 6$.

14. $\cdot 1x+3y=2\cdot 6$, $x-1\cdot 6y=10\cdot 2$.

15. $\cdot 05x+\cdot 03y=29$, $\cdot 03x-\cdot 04y=0$.

16. $x=\frac{y-1}{3}=\frac{x-9}{2}$.

17. $\frac{x}{5}-\frac{y}{3}=\frac{x}{7}-\frac{y}{6}=3$.

18. $\frac{x-y}{2}=\frac{2x}{5}=6-y$.

19. $x+\frac{y}{6}=y+\frac{x}{6}=7$.

20. $\frac{x}{3}-\frac{y}{4}=3x+7y+26=6$.

21.* $\frac{x-y}{2}+\frac{x+y}{3}=2\frac{1}{3}$,
$\frac{x+y}{2}+\frac{x-y}{3}=4\frac{1}{3}$.

22. $x+\frac{1}{2}y-\frac{1}{2}=y+\frac{1}{3}(x+y)$,
$2y-x+1=\frac{1}{3}(2x+y+3)$.

23. $x+\frac{1}{3}y=y-2$,
$y+\frac{1}{4}x=x+6$.

24. $5(x+y)-7(x-y)=26$,
$(3x+7y)\div 4=(6x-y)\div 3$.

25. $8x-7y=12$,
$\frac{x-2y}{4}+\frac{2x-y}{3}=1$.

26. $\frac{x+1}{10}=\frac{3y-5}{2}=\frac{x-y}{8}$.

27. $\frac{3}{4}y-\frac{1}{2}x+24=\frac{5}{7}y+\frac{1}{3}x+11=0$.

EXERCISE 42

Solve, by using two unknowns, and verify:

1. The sum of two numbers is 40 and their difference is 12. Find the numbers.

2. The sum of two numbers is 19. The sum of 3 times the first and 4 times the second is 64. Find the numbers.

3. If 4 lb. of tea and 7 lb. of sugar cost \$2·42, and 5 lb. of tea and 3 lb. of sugar cost \$2·68, find the cost of each per lb.

4. Find two numbers such that 7 times the first is greater than twice the second by 23, and 5 times the first and 3 times the second make 136.

5. If 5 horses and 6 cows cost \$840, and 3 horses and 2 cows cost \$440, find the cost of a horse.

6. If either 9 tables and 7 chairs, or 10 tables and 2 chairs, can be bought for \$156, what is the cost of each?

7. If 3 men and 4 women earn \$164 in 4 days and 5 men and 2 women earn \$135 in 3 days, find the daily wages of a man and of a woman.

8. Find two numbers such that $\frac{1}{2}$ of the first and $\frac{1}{3}$ of the second is 26, and $\frac{1}{3}$ of the first and $\frac{1}{4}$ of the second is 8.

9. Three bushels of wheat cost 20 cents more than 5 bushels of corn, and 2 bushels of wheat and 1 bushel of corn cost \$2·30. What is the price of each per bushel?

10. In 10 years a man will be twice as old as his son, but 8 years ago the man was 8 times as old as his son. Find their present ages.

11. If the sum of two numbers be added to 3 times their difference the result is 18; if twice the sum be added to their difference the result is 26. Find them.

12. A merchant sells 33 suits, some at $35 each and the others at $25, and receives $945. How many did he sell at each price?

13. Find two numbers such that 5% of the first is greater than 6% of the second by 3, and 7% of the second is greater than 4% of the first by 7·5.

14. If 3 algebras and 4 arithmetics cost $2·95, and 2 algebras and 3 arithmetics cost $2·10, find the cost of 6 algebras and 2 arithmetics.

15. A bull's eye counts 5 and an inner 4. In 10 shots a marksman scores 46 points, each shot being either a bull's eye or an inner. How many of each kind did he make?

16. A classroom has 25 seats, some double and some single. If there is seating accommodation for 42 pupils, how many double seats are there?

17. A man bought 8 cows and 50 sheep for $900. He sold the cows at a gain of 20% and the sheep at a gain of 10%, and received in all $1030. Find the cost of a cow?

18. If 10 men and 8 boys receive $37, and 4 men receive $1 more than 6 boys, how much does each boy receive?

19. A man bought 20 bushels of wheat and 15 bushels of corn for $36 and 15 bushels of wheat and 25 bushels of corn, at the same rate, for $32·50. How much did he pay per bushel for each?

20. Find two numbers such that, if the first be increased by 8 it will be twice the second, and if the second be increased by 31 it will be three times the first.

21. A farmer bought 100 acres of land for $4220, part at $37 and the rest at $45 per acre. How many acres were there of each kind?

22. Find two numbers such that 7 times the greater and 5 times the less together make 332, and 51 times their difference is 408.

23. The quotient is 20 when the sum of two numbers is divided by 3, and the quotient is 7 when their difference is divided by 2. Find the numbers.

24. A grocer bought tea at 60c. a lb. and coffee at 30c., the total cost being \$96. He sold the tea at 75c. a lb. and the coffee at 35c., and gained \$21. How many lb. of each did he buy ?

25. Three times the greater of two numbers exceeds twice the less by 90, and twice the greater together with three times the less is 255. Find the numbers.

26. The sum of two fractions whose denominators are 2 and 5 respectively is 2·9. If the numerators be interchanged the sum would be 4·1. Find the fractions.

27. Divide 142 into two parts so that when the larger part is divided by 17 and the other by 19 the sum of the quotients will be 8.

28. A farm was rented for \$650, part of it at \$6 and the rest at \$8 per acre. If the rates had been interchanged the rental would have been \$750. How many acres were in the farm ?

29. *A*'s age 3 years ago was half of *B*'s present age. In 7 years the sum of their ages will be 77 years. Find their present ages.

30. A man travelled 240 miles in 4 days, diminishing his rate each day by the same distance. The first two days he went 136 miles. How far did he go each day ?

EXERCISE 44 (Review of Chapter VII)

1. Solve $2x+3y=38$, $3x+2y=37$.

2. I fire 20 shots at a target. If a hit counts 5 and a miss counts -2, how many hits did I make if my net score is 51 ?

3. Solve $7x-2y=13$, $2x+3y=43$.

4. The average marks of those who passed an examination was 65, and of those who failed was 25. If there were 1000 candidates in all and their average was 53, how many passed ?

5. Solve $2(x-y)=3(x-4y)$, $14(x+y)=11(x+8)$.

6. At an election *A*'s majority was 384, which was $\frac{3}{11}$ of the whole number of votes. How many votes did *A* receive ?

7. Solve $\frac{1}{2}(x+5)-5=\frac{1}{3}(y+2)$, $\frac{1}{3}(y+8)-3=\frac{1}{4}(x-3)$.

8. Divide \$5600 into two parts, so that the income from one part at 3% may be equal to the income on the other part at 4%.

9. Solve $\frac{x}{3}+\frac{y}{4}=3x-7y-37=0$.

10. Two numbers differ by 11, and ¼ of the larger is 1 more than ⅓ of the smaller. Find the numbers.

11. If $px+qy$ is 74 when $p=5$ and $q=3$, and is 76 when $p=6$ and $q=2$, find x and y.

12. If 3% of *A*'s salary plus 4% of *B*'s salary is $93, and 5% of *A*'s plus 3% of *B*'s is $111, find their salaries.

13. Solve $21y+20x=165$, $77y-30x=295$.

14. Divide 100 into two parts so that ⅕ of the greater part exceeds ¼ of the less by 2.

15. Solve $5x-2y=7x+2y=x+y+11$.

16. If 3 men and 4 boys earn $26, and 5 men and 2 boys earn $34, what would 7 men and 3 boys earn ?

17. Solve $\frac{1}{2}(x+1)-\frac{1}{3}(y+2)=3$, $\frac{1}{3}(x+2)+\frac{1}{4}(y+3)=4$.

18. If $3x-4=ax+b$ when $x=2$ and when $x=5$, show that $a=3$ and $b=-4$.

19. I bought a horse and carriage for $400. I sold the horse at a profit of 20% and the carriage at a loss of 4%, and on the whole transaction I gained 5%. What did each cost ?

20. Solve $\dfrac{3x}{2}-2y=2x-\dfrac{3y}{2}=7$.

21. A man pays a debt of $52 in $5 bills and $1 bills. If the number of bills is 24, how many are there of each ?

22. Solve $19x-21y=100$, $21x-19y=140$.

23. *A*'s wages are half as high again as *B*'s, but *A* spends twice as much as *B*. If *A* saves $5 and *B* $10 per week, what are the wages of each per week ?

24. If $23x+15y=91$, and y is 50% more than x, find x and y.

25. When a man was married his age was ¼ more than his wife's age. His age 8 years afterwards was ⅕ more than his wife's age. How old was he when he was married ?

26. If $3(5x-2y)=2(3x+6y)$, find x in terms of y.

27. A man has two farms rented at $5 per acre and the total rent is $1100. When the rent of the first is reduced 20% and the second is increased 20%, the total rent is $1120. How many acres are there in each ?

28. If $\frac{x}{3} + \frac{y}{4} = \frac{x}{6} + \frac{11y}{16} = 9$, find the value of $\frac{x}{7} + \frac{y}{2}$.

29. Seven years ago B was three times as old as A, but in 5 years he will be only twice as old. What are their present ages?

30. Solve $\frac{x}{2} + \frac{y}{2\frac{1}{3}} = 1\frac{1}{2}$, $\frac{x-y}{11} = \frac{x}{1\frac{3}{4}} - \frac{3}{14}$.

31. Solve $\frac{3x+5}{2} = 5 + \frac{y+1}{4}$, $\frac{y+8}{3} = 3 + \frac{x+5}{4}$.

CHAPTER VIII

TYPE PRODUCTS AND SIMPLE FACTORING

58. Factor. When a quantity is the product of two or more quantities, each of the latter is called a **factor** of the given quantity.

Thus, the factors of $3bc$ are 3, b and c.
The product of $b+c$ and a is $ab+ac$,
$\therefore$ the factors of $ab+ac$ are a and $b+c$,
or $$ab+ac=a(b+c).$$
Similarly, $$ab-ac=a(b-c).$$

When $x+y+z$ is multiplied by a, the product $ax+ay+az$ contains the factor a in each term.

If we wish to factor $ax+ay+az$, we recognize that since a is a factor of each term, it must be a factor of the whole expression. The remaining factor is the quotient found by dividing the expression by a.

$$a\overline{)ax+ay+az}$$
$$x+\ y+\ z$$

Then $$ax+ay+az=a(x+y+z).$$

This is seen to be similar to the method in arithmetic. If we wish to factor 485, we see that 5 is a factor. How do we obtain the other factor ?

Ex.—Factor $4a^2-6ab$.

Here we see that 2 and a are factors of each term and therefore $2a$ is a factor. On division the other factor is $2a-3b$.

$$\therefore\ 4a^2-6ab=2a(2a-3b).$$
Similarly, $$3bx+6cx=3x(\qquad\quad).$$
$$ab-a^2-a^3=a(\qquad\quad).$$

The result of the factoring may be verified by multiplication and this may usually be done mentally.

EXERCISE 45

Fill in the blanks in the following:

1. $4x+6=2(\quad)$.
2. $3a-9=3(\quad)$.
3. $5x-10y=5(\quad)$.
4. $ax+3x=x(\quad)$.
5. $bx-by=b(\quad)$.
6. $x^2+x=x(\quad)$.
7. $7p^2-6p=p(\quad)$.
8. $6y^2+3y=3y(\quad)$.
9. $8x^3-2x^2=2x^2(\quad)$.

Factor the following and verify:

10. $2y+4$.
11. $6m-12$.
12. $3x^2-15$.
13. $ab+ac$.
14. $am-bm$.
15. $ab+ac+a$.
16. $mx+my-mz$.
17. x^2-7x.
18. $5a^2+10ab$.
19. $4x^3+6x^2+2x$.
20. $a^2x+a^2y-a^3$.
21. $15x^2-10xy$.
22. $2ax-4ay+6az$.
23. $x^3-3x^2y+xy^2$.
24. $4ab+6a^2b^2-8abc$
25. $(x+y)a+(x+y)b$.
26. $x(a-b)+y(a-b)$.
27. $2x(b-c)-2(b-c)$.

59. Definition. An algebraic expression containing only one term is called a **monomial**, one of two terms is called a **binomial**, one of three terms a **trinomial**, and one of more than three terms a **multinomial** or **polynomial**.

Thus, $2x-5$ is a binomial and a^2+3a+7 is a trinomial.

60. Product of two Binomials. The pupil should be able to write down mentally the product of two simple binomials like $x+2$ and $x+3$.

$$\begin{array}{r} x+2 \\ x+3 \\ \hline x^2+2x \\ +3x+6 \\ \hline x^2+5x+6 \end{array}$$

What is the source of the first term (x^2) in the product? What is the source of the last term (6)? What two quantities were added to give the middle term ($5x$)? How were these two quantities obtained?

In the product of $x+1$ and $x+7$, what would be the first term, the last term, the middle term? What is the complete product?

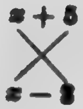
In the product of $x-2$ and $x-3$, what is the first term, last term, middle term? How does the product differ from the product of $x+2$ and $x+3$?

EX.—Multiply $x-5$ by $x+3$.

Why is the last term negative? The middle term is the sum of $+3x$ and $-5x$ or $-2x$. What is the complete product?

What is the middle term in the product of $x+5$ and $x-3$?

The middle term in every case is seen to be the sum of the two cross products, each taken with the proper sign.



EXERCISE 48 (1-22, Oral)

State the products of:

1. $\begin{array}{l} x+1 \\ x+2 \\ \hline \end{array}$ 2. $\begin{array}{l} x+5 \\ x+11 \\ \hline \end{array}$ 3. $\begin{array}{l} x-3 \\ x-4 \\ \hline \end{array}$ 4. $\begin{array}{l} x-5 \\ x-12 \\ \hline \end{array}$

5. $\begin{array}{l} y-6 \\ y+5 \\ \hline \end{array}$ 6. $\begin{array}{l} m-2 \\ m+4 \\ \hline \end{array}$ 7. $\begin{array}{l} a+b \\ a+2b \\ \hline \end{array}$ 8. $\begin{array}{l} x-3y \\ x-2y \\ \hline \end{array}$

9. $\begin{array}{l} x+4y \\ x-3y \\ \hline \end{array}$ 10. $\begin{array}{l} y-5x \\ y+5x \\ \hline \end{array}$ 11. $\begin{array}{l} p-6q \\ p+11q \\ \hline \end{array}$ 12. $\begin{array}{l} a-2 \\ a-\frac{1}{2} \\ \hline \end{array}$

13. $\begin{array}{l} ab-1 \\ ab-3 \\ \hline \end{array}$ 14. $\begin{array}{l} xy-7 \\ xy+7 \\ \hline \end{array}$ 15. $\begin{array}{l} pq-r \\ pq+r \\ \hline \end{array}$ 16. $\begin{array}{l} ax-2by \\ ax-3by \\ \hline \end{array}$

17. $(a+2)(a+1)$. 18. $(x-y)(x-4y)$. 19. $(p+q)(p-q)$.

20. $(x-3y)(x+2y)$. 21. $(m+4n)(m-5n)$. 22. $(b-3)(b-\frac{1}{2})$.

Remove the brackets, simplify and check:

23. $3(x+2)+2(3x-1)-(x-3)$.

24. $(x+1)(x+2)+(x+2)(x+3)$.

25. $(y+3)(y-2)+(y-5)(y+4)$.

26. $(x+1)^2+(x-1)(x+1)+(x+1)(x-2)$.

27. $2(m+1)(m+2)+3(m-1)(m-2)$.

28. $4(x+3)(x+1)-(x+1)(x+12)$.

61. Factors of Trinomials. The product of two binomials, like those in the preceding exercise, is seen to be a trinomial.

To find the factors of a trinomial we must reverse the process of multiplication.

Ex. 1.—Factor x^2+6x+8.

Since the last term is positive, the last terms in the factors must have like signs, and since the middle term is positive, the signs must both be plus.

∴ the factors are of the form $(x+\quad)(x+\quad)$.

The last terms in the factors must be factors of 8, so they must be 1 and 8 or 2 and 4.

$$\begin{array}{c} x+1\\ x+8\\ \hline \end{array}\qquad\qquad \begin{array}{c} x+2\\ x+4\\ \hline \end{array}$$

Which of these when multiplied will give the proper middle term?

What are the factors of x^2-6x+8?

The factors of $x^2-9x+14$ must be of the form $(x-\quad)(x-\quad)$. What are the factors?

Ex. 2.—Factor $x^2-2x-15$.

Here the factors must be of the form $(x-\quad)(x+\quad)$, since -15 must be the product of two numbers differing in sign. The possible combinations are:

$$\begin{array}{r} x-15\\ x+\ 1\\ \hline \end{array}\qquad \begin{array}{r} x+15\\ x-\ 1\\ \hline \end{array}\qquad \begin{array}{r} x-5\\ x+3\\ \hline \end{array}\qquad \begin{array}{r} x+5\\ x-3\\ \hline \end{array}$$

Which of these sets of factors is the correct one?

In factoring a trinomial like $x^2-8x+15$, we require two factors of 15 whose algebraic sum is -8. They are evidently -5 and -3.

$$\therefore x^2-8x+15=(x-5)(x-3).$$

In factoring $x^2-4x-21$, we require two factors of -21 whose algebraic sum is -4, and they are evidently -7 and 3.

$$\therefore x^2-4x-21=(x-7)(x+3).$$

The pupil is advised to write the factors under each other, below the expression he is attempting to factor.

Thus,

$$\begin{array}{l|l} \quad x^2-6x-16 & \quad x^2+11xy-42y^2\\ \quad x\qquad -\ 8 & \quad x\qquad +14y\\ \quad x\qquad +\ 2 & \quad x\qquad -\ 3y\\ \therefore x^2-6x-16=(x-8)(x+2). & \therefore x^2+11xy-42y^2=(x+14y)(x-3y). \end{array}$$

EXERCISE 47 (1-15, Oral)

Factor:

1. x^2+8x+7.
2. x^2+6x+5.
3. $y^2+8y+15$.
4. $a^2+22a+21$.
5. $x^2+8x+12$.
6. $b^2+10b+24$.
7. $a^2+3ab+2b^2$.
8. $m^2+7mn+10n^2$.
9. $y^2+40xy+39x^2$.
10. x^2-5x+6.
11. x^2-7x+6.
12. $x^2-12x+11$.
13. $x^2-4xy+3y^2$.
14. $a^2-11ab+28b^2$.
15. $m^2-7mn+12n^2$.

16. x^2-x-20.   17. y^2-y-30.   18. a^2+a-30.

19. $x^2-5x-14$.   20. $m^2-6m-40$.   21. $x^2-10x-24$.

22. $a^2b^2+8ab+15$.   23. $x^2y^2-11xy+30$.   24. x^4-10x^2+9.

25. a^2+6a+9.   26. $x^2-14x+49$.   27. y^4-12y^2+36.

Use factoring to simplify the following:

28.* $\dfrac{a^2+5a+4}{a+4}+\dfrac{a^2+4a-5}{a+5}$.   29. $\dfrac{m^2-5m+6}{m-3}-\dfrac{m^2-7m+12}{m-4}$.

30. $\dfrac{(x^2+3x+2)(x-5)}{x^2-3x-10}$.   31. $\dfrac{3x^2-6x}{3x}+\dfrac{2x^3-4x^2}{2x^2}+\dfrac{x^2-5x+4}{x-1}$.

32. What factor is common to

(1) x^2-x-30 and $x^2-2x-35$?
(2) a^2+ab and $a^2+3ab+2b^2$?

Find three factors of:

33. $2x^2-10x+12$.   34. $3a^2+3a-36$.   35. x^3-8x^2+7x.

36. If the expression x^2+mx-6 has two binomial factors with integral coefficients, what are all the possible values of m?

37. Is the expression $x^2-3x-10$ factored when it is written in the form $x(x-3)-10$?

62. Square Root of a Monomial. When a number is the product of two equal factors, each factor is called a **square root** of the number.

Thus, $16=4\times 4$, therefore a square root of 16 is 4.
But $16=-4\times -4$, therefore a square root of 16 is also -4.
Similarly, the square root of 25 is $+5$ or -5,
and the square root of $9a^2$ is $+3a$ or $-3a$.

Thus it is seen that every number has **two square roots** differing only in sign.

It is customary to call the positive square root of a number the **principal square root**.

63. Radical Sign. The symbol $\sqrt{}$, called the **radical sign** or **root sign**, is used to indicate the principal square root of a number.

Thus, $\sqrt{25}=5$, $\sqrt{a^2}=a$, $\sqrt{9x^2y^2}=3xy$.

When both the positive and negative square roots are considered, both signs must precede the radical sign.

Thus, $\sqrt{9}=3$ not -3; $-\sqrt{9}=-3$ not $+3$, but $\pm\sqrt{9}=\pm 3$, and is read "plus or minus the square root of 9 equals plus or minus 3."

Thus, $$\sqrt{4}+\sqrt{9}=2+3=5,$$
but $$\pm\sqrt{4}\pm\sqrt{9}=\pm 2\pm 3=\pm 5 \text{ or } \pm 1.$$

If we represent the square root of 16 by x, then $x^2=16$. To solve this equation, take the square root of each side,

$$\therefore\ x=\pm 4.$$

We might have said $\pm x=\pm 4$, which includes the four statements:

$$+x=+4,\quad +x=-4,\quad -x=+4,\quad -x=-4.$$

If both terms of the last two be multiplied by -1, the statements become the same as the first two, which are represented by $x=\pm 4$.

We see then, that *it is necessary to attach the double sign to the square root of only one side of the equation.*

Ex.—Solve $(x+1)^2=25$.

Take the square root of each side,

$$\therefore\ x+1=\pm 5,$$
$$\therefore\ x=\pm 5-1=5-1 \text{ or } -5-1,$$
$$=4 \text{ or } -6.$$

Show by substitution that each root satisfies the given equation.

EXERCISE 48 (1-16, Oral)

State the two square roots of:

1. 36.
2. 81.
3. 121.
4. $2\frac{1}{4}$.
5. y^2.
6. b^2c^2.
7. $25a^2$
8. $64x^2y^2$.
9. $\frac{1}{4}a^2$.
10. $\frac{1}{9}m^2n^2$.
11. $\frac{4}{25}p^4$.
12. $6\frac{1}{4}x^2$.

Solve the following equations:

13. $x^2=9$.
14. $3x^2=75$.
15. $x^2=4a^2$.
16. $x^2=a^2b^2$.
17.* $(x+2)^2=81$.
18. $(x-3)^2=49$.
19. $(x-5)^2=0$.

20. If the area of a square is 100 square inches, find the length of its side.

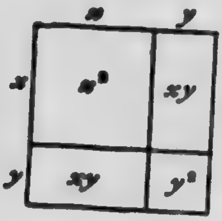
21. If r is the radius of a circle the area is πr^2, where $\pi = 3\frac{1}{7}$ approximately. If the area of a circle is 154 square inches, what is the radius, or what is the value of r, if $3\frac{1}{7}r^2 = 154$?

22. Find the radius of a circle whose area is 616 sq. in.

23. If r is the radius of a sphere the area of its surface is given by the formula, area $= 4\pi r^2$. If the area of the surface of a sphere is 154 sq. in., what is the radius?

64. Squares of Binomials. If we multiply $x+y$ by $x+y$, the result, which will be the square of $x+y$, is $x^2+2xy+y^2$.

$$\therefore \quad (x+y)^2 = x^2+2xy+y^2.$$



The diagram shows a geometrical illustration of this identity.

The first and last terms in $x^2+2xy+y^2$ are the squares of the terms of $x+y$, and the middle term is *twice* the product of x and y.

Therefore, *the square of the sum of two numbers is equal to the sum of the squares of the numbers*, **increased** *by twice their product.*

Also
$$(x-y)^2 = x^2-2xy+y^2.$$

Therefore, *the square of the difference of two numbers is equal to the sum of the squares of the numbers* **decreased** *by twice their product.*

In the square of a sum all the terms are positive, and in the square of a difference the middle term is negative.

Thus,
$$\begin{aligned}(3a+2b)^2 &= (3a)^2+2(3a)(2b)+(2b)^2,\\ &= 9a^2 + 12ab + 4b^2.\\ (5x-3y)^2 &= (5x)^2-2(5x)(3y)+(3y)^2\\ &= 25x^2 - 30xy + 9y^2.\\ (\tfrac{1}{2}x-4y)^2 &= (\tfrac{1}{2}x)^2-2(\tfrac{1}{2}x)(4y)+(4y)^2,\\ &= \tfrac{1}{4}x^2 - 4xy + 16y^2.\end{aligned}$$

EXERCISE 49 (1-16, Oral)

What are the squares of :

1. $a+1$.
2. $y+2$.
3. $m-1$.
4. $x-4$.
5. $2a+1$.
6. $1-3x$.
7. $p-q$.
8. $2x+3$.
9. $2a-3$.
10. $m-2n$.
11. $3x-2y$.
12. $4x-3a$.
13. $\frac{1}{2}-x$.
14. $2y-\frac{1}{2}$.
15. $3x-\frac{1}{3}$.
16. $-x-2$

Simplify :

17.* $(x+1)^2+(x-1)^2$.

18. $(a-b)^2+(a+b)^2$.

19. $(2x+1)^2+(x-2)^2$.

20. $(a+b)^2-(a-b)^2$.

21. $(3m-n)^2-(2m+n)^2$.

22. $(3x+2y)^2-(2x-3y)^2$.

23. $(x+1)^2+(x+2)^2+(x+3)^2$.

24. $x-1)^2+(x-2)^2-(x-3)^2$.

25. $2(a+1)^2+3(a-1)^2-5(a-2)^2$.

26. Find the value of $a^2+b^2+c^2$ when $a=x-y$, $b=x+y$, $c=x-2y$.

27. Simplify $(x+1)^2+(x-2)^2+(x-3)^2-3(x-4)^2$.

28. From the sum of the squares of $x+2$, $x+3$, $x+4$, subtract the sum of the squares of $x-2$, $x-3$, $x-4$.

29. Simplify $(2a-3b)^2+(3a+2b)^2-(2a+2b)^2$.

30. If two numbers differ by 2, show that the difference of their squares is equal to twice their sum.

31. By how much does the square of $x+\frac{2}{x}$ exceed the square of $x-\frac{2}{x}$?

32. Show that the sum of the squares of three consecutive numbers is greater by two than three times the square of the middle number.

33. The square of 1234 is 1,522,756. Find the square of 1235.

34. The square of $2\frac{1}{2}=2\times3+\frac{1}{4}=6\frac{1}{4}$; the square of $5\frac{1}{2}=5\times6+\frac{1}{4}=30\frac{1}{4}$, etc. In the same way find the squares of $6\frac{1}{2}$, $8\frac{1}{2}$, $20\frac{1}{2}$. Prove that this method may be used to find the square of any number ending in $\frac{1}{2}$. (Let the number be $n+\frac{1}{2}$.)

65. Square Roots of Trinomials. Any trinomial which is of the form $a^2+2ab+b^2$ or $a^2-2ab+b^2$ is a perfect square.

In order that a trinomial may be a square, the first and last terms must each be a square and the middle term must be twice the product of the quantities which were squared to produce the first and last terms.

Thus, $9x^2+24xy+16y^2$ is a square, because

(1) $9x^2$ is the square of $3x$,
(2) $16y^2$ is the square of $4y$,
(3) $24xy$ is twice the product of $3x$ and $4y$.

$\therefore$ $9x^2+24xy+16y^2=(3x+4y)^2$.

$\therefore$ the square root of $9x^2+24xy+16y^2$ is $3x+4y$.

Is $4m^2-12mn+9n^2$ a perfect square? What is it the square of? What is its square root?

Similarly,

$$25x^2-10x+1=(5x-1)^2,$$
$$36x^2+24x+4=(\qquad)^2,$$
$$a^2b^2-6ab+9=(\qquad)^2.$$

Why is $a^2+5ab+25b^2$ not a square? Is it the square of $a+5b$? How would you change it so that it would be a square?

The square root of $a^2+2ab+b^2$ is $a+b$, but $-(a+b)$ or $-a-b$ is also a square root, since

$$(-a-b)^2=a^2+2ab+b^2.$$

It is customary, however, in stating the square root of a trinomial to give only that one which has its first term positive.

EXERCISE 30 (1-24, Oral)

Express as squares:

1. $x^2+2xy+y^2$.
2. y^2-2y+1.
3. $4x^2+4x+1$.
4. $4a^2+20a+25$.
5. $9a^2-24a+16$.
6. $16x^2-8x+1$.
7. $a^2b^2-2ab+1$.
8. $1-6y+9y^2$.
9. $9x^2-18xy+9y^2$.
10. $a^2b^2c^2-2abc+1$.
11. $x^2+x+\frac{1}{4}$.
12. $y^2-xy+\frac{1}{4}x^2$.

What is the square root of:

13. $9a^2+12a+4$.
14. $x^2-4xy+4y^2$.
15. $1-6x+9x^2$.
16. $4a^2b^2-20ab+25$.
17. $4m^2+2m+\frac{1}{4}$.
18. $a^2-14ab+49b^2$.
19. $4-4a+a^2$.
20. $9-12x+4x^2$.
21. $9x^2-30xy+25y^2$.

Supply the missing terms, so that the following will be perfect squares:

22. $a^2+\ .\ .\ .+b^2$.
23. $x^2-\ .\ .\ .+4y^2$.
24. $x^2+6x\ .\ .\ .\ .$
25. $4m^2-\ .\ .\ .+9$.
26. $9a^2+18a\ .\ .\ .$
27. $.\ .\ .-4xy+4y^2$.

28. If $16a^2-ma+4$ is a perfect square, what is the value of m? Give two answers and verify each.

29. What is the square root of $9x^2+6x+1$? Check by putting $x=10$.

30. Solve the equations and verify:

(1) $\sqrt{x^2+2x+1}+\sqrt{x^2+10x+25}=14.$

(2) $3\sqrt{x^2-4x+4}-2\sqrt{x^2+6x+9}=-2.$

(3) $\sqrt{9x^2+6x+1}+\sqrt{4x^2+4x+1}+\sqrt{x^2-2x+1}=13.$

31. Show that

$$2\sqrt{a^2-6a+9}-\sqrt{a^2-4a+4}=3\sqrt{a^2-2a+1}-\sqrt{4a^2+4a+1}.$$

66. Product of the Sum and Difference. The product of $a+b$ and $a-b$ is a^2-b^2,

$$\therefore\quad (a+b)(a-b)=a^2-b^2.$$

$$\begin{array}{ll} a & +b \\ a & -b \\ \hline a^2+ab & \\ \quad -ab & -b^2 \\ \hline a^2 & -b^2 \end{array}$$

Here the two factors multiplied are the sum and difference of the same two quantities a and b, and the product is the difference of the squares of a and b.

Therefore, *the product of the sum and difference of the same two quantities is equal to the difference of their squares.*

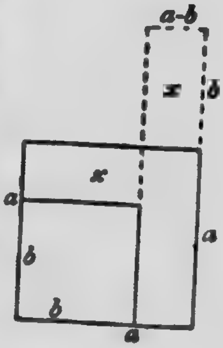
Thus,

$$(x+y)(x-y)=x^2-y^2.$$
$$(2a+3b)(2a-3b)=(2a)^2-(3b)^2=4a^2-9b^2.$$
$$(3a^2-b)(3a^2+b)=(3a^2)^2-b^2=9a^4-b^2.$$
$$(\tfrac{1}{2}+3x)(\tfrac{1}{2}-3x)=(\tfrac{1}{2})^2-(3x)^2=\tfrac{1}{4}-9x^2.$$

67. Factors of the Difference of Two Squares.

Since $a^2-b^2=(a+b)(a-b)$, *the factors of the difference of two squares are the sum and the difference of the quantities squared.*

The diagram shows how this identity may be illustrated geometrically.



Thus, $9x^2-25y^2=(3x)^2-(5y)^2$, which shows that it is the difference of the squares of $3x$ and $5y$.

Therefore, one factor of $9x^2-25y^2$ is the sum of $3x$ and $5y$, and the other is the difference of $3x$ and $5y$.

That is, $$9x^2-25y^2=(3x+5y)(3x-5y).$$

Similarly, $$16m^2-9=(4m)^2-3^2=(4m+3)(4m-3).$$

If we wish to factor $8x^2-2y^2$, we should recognize that 2 is a factor of each term.

$$\therefore\quad 8x^2-2y^2=2(4x^2-y^2)=2(2x+y)(2x-y).$$

EXERCISE 51 (1-24, Oral)

State the products of:

1. $m+n,\ m-n.$
2. $p-q,\ p+q.$
3. $a+2,\ a-2.$
4. $x-5,\ x+5.$
5. $2a+1,\ 2a-1.$
6. $3x-2,\ 3x+2.$
7. $(2a-3x)(2a+3x).$
8. $(4x+5y)(4x-5y).$
9. $(x+\frac{1}{2})(x-\frac{1}{2}).$
10. $(x^2-2y)(x^2+2y).$
11. $(5x+ab)(5x-ab).$
12. $\left(2x-\frac{y}{3}\right)\left(2x+\frac{y}{3}\right).$

State the factors of:

13. $x^2-1.$
14. $y^2-4.$
15. $a^2-4b^2.$
16. $4m^2-n^2.$
17. $4p^2-9q^2.$
18. $x^2-\frac{1}{4}.$
19. $9-x^2.$
20. $1-16a^2b^2.$
21. $25-49x^2.$
22. $a^4-25.$
23. $a^2b^2-49.$
24. $99^2-98^2.$

Simplify:

25.* $(a-2)(a+2)+(2a-1)(2a+1).$

26. $(2a-3b)(2a+3b)-(a+b)(a-b).$

27. $2(x-3y)(x+3y)+2(3y-x)(3y+x).$

28. $2(p-q)^2+3(p+q)(p-q)-5(p+2q)(p-2q).$

29. Find the product of $x-a$, $x+a$ and x^2+a^2.

30. From the product of $x-1$, $x+1$ and x^2+1, subtract the product of $x-2$, $x+2$ and x^2+4.

Find three factors of:

31. $3x^2-3y^2.$
32. $5x^2-2$
33. $a^3-a.$
34. $mx^2-ma^2.$
35. $5-45p^2.$
36. $x^4-y^4.$
37. $\pi R^2-\pi r^2.$
38. $a(x^2-1)+b(x^2-1).$

39. Why is the difference between the squares of any two consecutive numbers always equal to their sum?

40. Simplify $(a^2-b^2)(a^2-5ab+6b^2)\div(a^2-3ab+2b^2)$.

41. Simplify $\dfrac{x^2-y^2}{x-y}+\dfrac{x^2-y^2}{x+y}$ and $\dfrac{x^2-16}{x-4}-\dfrac{x^2-9}{x+3}$.

42. Solve $\dfrac{x^2-1}{x+1}+\dfrac{x^2-9}{x-3}=10$; $2(x-5)(x+5)=15+(x-1)(x+1)$.

68. Numerical Applications of Products and Factors.

In this Chapter we have developed certain formulæ concerning products and factors.

(1) $(a-b)^2 \quad = a^2-2ab+b^2$.
(2) $(a+b)^2 \quad = a^2+2ab+b^2$.
(3) $(a+b)(a-b) = a^2-b^2$.

These formulæ are true for all values of the letters involved. By substituting particular numbers for the letters we will see how some arithmetical operations might be simplified.

(1) Since $(a-b)^2=a^2-2ab+b^2$,
$99^2 = (100-1)^2 = 10000-200+1 = 10001-200 = 9801.$
$37^2 = (40-3)^2 = 1600-240+9 = 1609-240 = 1369.$
$998^2 = (1000-2)^2 = \qquad = \qquad =$
$89^2 = (90-1)^2 = \qquad = \qquad =$

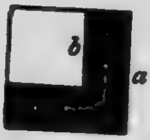
(2) Since $(a+b)^2=a^2+2ab+b^2$,
$92^2 = (90+2)^2 = 8100+360+4 = 8464.$
$121^2 = (120+1)^2 = 14400+240+1 = 14641.$
$75^2 = (70+5)^2 = \qquad =$

(3) Since $(a+b)(a-b)=a^2-b^2$,
$92\times 88 = (90+2)(90-2) = 90^2-2^2 = 8100-4 = 8096.$
$65\times 75 = (70-5)(70+5) = 70^2-5^2 = 4900-25 = 4875.$
$27\times 23 = (25+2)(25-2) = \qquad = \qquad =$
$87\times 93 = (\qquad)(\qquad) = \qquad = \qquad =$

(4) Since $a^2-b^2=(a+b)(a-b)$,
$53^2-52^2 = (53+52)(53-52) = 105\times 1 = 105.$
$41^2-31^2 = (41+31)(41-31) = 72\times 10 = 720.$
$727^2-627^2 = (\qquad)(\qquad) = \qquad =$
$67^2-33^2 = (\qquad)(\qquad) = \qquad =$

69. Some Geometrical Applications.

(1) If a is the length of the side of the large square and b the side of the small square the area of the shaded portion is evidently a^2-b^2.



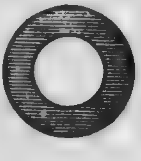
If we wish to find the area of the shaded part when $a=77$ and $b=23$, we have

$$a^2-b^2=77^2-23^2=(77+23)(77-23)=100\times 54=5400.$$

If $a=225$ and $b=125$, find the difference in the areas of the two squares.

(2) The radius of the large circle is R and of the small circle is r. The area of the large circle is $\frac{22}{7}R^2$ and of the small one is $\frac{22}{7}r^2$,

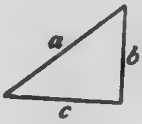
$\therefore$ the area of the shaded part is $\frac{22}{7}(R^2-r^2)$.



If $R=39$ and $r=31$, find the area of the ring.

$$\begin{aligned}\text{The area}\quad &=\tfrac{22}{7}(R^2-r^2)=\tfrac{22}{7}(39^2-31^2)\\ &=\tfrac{22}{7}(39+31)(39-31)=\tfrac{22}{7}\times 70\times 8=1760.\end{aligned}$$

If $R=89$ and $r=82$, show that the area of the ring is 3762.



(3) In the right-angled triangle in the figure it is shown in geometry that

$$b^2+c^2=a^2 \text{ or } b^2=a^2-c^2 \text{ or } c^2=a^2-b^2.$$

If $a=41$ and $c=40$, find the length of b.

$$b^2=a^2-c^2=41^2-40^2=81\times 1=81,$$

$$\therefore\quad b=\sqrt{81}=9.$$

If $a=61$ and $b=11$, show that $c=60$.

EXERCISE 52

Use short methods in the following:

1. Find the squares of 98, 999, 119, 58, 799.
2. Find the products of 91 × 89, 61 × 59, 47 × 53, 203 × 197.
3. Find the values of 52^2-48^2, 79^2-78^2, 215^2-205^2, 725^2-275^2, 673^2-573^2.
4. If $x^2=b^2-c^2$, find x when $b=13$, $c=12$; when $b=25$, $c=24$.
5. If $7x^2=64^2-57^2$, find the value of x.

6. Find the difference of the areas of squares whose sides are a and b for the following values:

$a=$	41	13	29	83	$15m$	2·85
$b=$	40	12	21	17	$14m$	2·15
$a^2-b^2=$						

7. Find the difference in the areas of circles whose radii are R and r for the following values:

$R=$	4	14	25	51	$19a$	3·25
$r=$	3	7	24	44	$5a$	2·35
$3\frac{1}{7}(R^2-r^2)=$						

EXERCISE 53 (Review of Chapter VIII)

Factor:

1. $3x+6y$.
2. $4m-12n$.
3. $ax-bx$.
4. $6ac-3bc$.
5. x^2+4x+4.
6. a^2-2a+1.
7. $4a^2-9x^2$.
8. x^2-3x+2.
9. $y^2-y-110$.
10. $2a^2-18$.
11. $100p^2-81q^2$.
12. $a^2-19a-20$.
13. $2a^2+6a+4$.
14. $300-3x^2$.
15. $(x+y)^2-1$.
16. a^4-b^4.

17. Solve $x^2=100$; $x^2=\frac{1}{4}$; $9x^2=4$; $5x^2=1{\cdot}25$

18. Write down the squares of:

$$2x-3,\ 5x-6,\ 4x-3y,\ a-\tfrac{1}{2},\ bc-\tfrac{1}{2}.$$

19. What are the square roots of: a^2+6a+9, $p^2-8p+16$, $m^2n^2-10mn+25$, $a^2-a+\frac{1}{4}$, $16x^2-40xy+25y^2$?

20. How much must be added to the middle term of $4a^2+3a+9$ to make it the square of $2a+3$?

21. What middle term must be inserted in $9x^2 \ldots +25y^2$ to make it a complete square? Give two answers.

22.* Find three factors of x^3-x, $3x^2-12$, a^3-3a^2+2a.

23. Solve $(x-3)^2=25$; $4(x-\frac{1}{2})^2=9$.

24. If $a=\pi r^2$, find r when $a=12{\cdot}56$, $\pi=3{\cdot}14$.

25. Find the values of 997^2, 875^2-75^2, 97×103, 81×81, 86×94, using algebraic methods.

26. Find four factors of $2x^4-32$, a^4-13a^2+36, $2m^3-18m$ and $a^2(x^2-y^2)-b^2(x^2-y^2)$.

27. Simplify $(b+1)^2+(b-1)^2+(c+1)^2+(c-1)^2$.

28. Simplify $(x+y)(x-y)+(x+2y)(x-y)+(x+y)(x-2y)$.

29. If $a=92$ and $b=88$, find the values of ab, a^2-b^2, a^2+b^2, using algebraic methods.

30. Simplify $(a-2b)(a+2b)+(a-4b)(a+4b)-2(a-3b)^2$.

31. What are all the possible values of b, if $x^2+bx+42$ is the product of two factors with positive integral coefficients ?

32. Simplify $\dfrac{x^2-y^2}{x-y}+\dfrac{x^2-4y^2}{x-2y}+\dfrac{x^2-9y^2}{x-3y}$.

33. If the square of 426 is 181476, find the squares of 427 and 425.

CHAPTER IX

SIMPLE APPLICATIONS OF FACTORING

70. Highest Common Factor. When a factor divides two or more expressions it is called a common factor of those expressions.

Thus, 4 is a common factor of 8, 12 and 20,
and a is a common factor of a^2, $2a$ and $3ab$.

As in arithmetic, the **highest common factor** (H.C.F.) is the product of all the simple common factors.

Thus, the simple common factors of $3a^2b$, $6ab^2$ and $9abc$ are 3, a and b, and therefore the H.C.F. is $3ab$.

In the case of monomials the H.C.F. may be written down by inspection.

Ex. 1.—Find the H.C.F. of $6m^2n$, $12m^2n^2$ and $9m^3n^3$.

(1) The H.C.F. of 6, 12 and 9 is 3.
(2) The highest power of m which is common is m^2.
(3) The highest power of n which is common is n.

$$\therefore \text{ the H.C.F. is } 3\times m^2\times n \text{ or } 3m^2n.$$

If the expressions are not monomials they must be factored when possible, after which the H.C.F. may be written down by inspection.

Ex. 2.—Find the H.C.F. of a^2+ab, $ab+b^2$, $a^2+3ab+2b^2$.

$$a^2+ab=a(a+b),$$
$$ab+b^2=b(a+b),$$
$$a^2+3ab+2b^2=(a+b)(a+2b),$$
$$\therefore \quad \text{the H.C.F.}=a+b.$$

EXERCISE 54 (1-12, Oral)

Find the H.C.F. of :

1. $3, 9, 12.$ 2. $16, 24, 40.$ 3. $2a, 4b, 8c.$

4. $3x, 6x, 12x.$ 5. $4ax, 6ax, 2x^2.$ 6. $a^2b, ab^2, a^2b^2.$

7. $3x^2, 4x^3, 5x^4.$ 8. $5a^2, 10a^3, 15a.$ 9. $17x^2y^2, 34x^2y, 51x^3.$

10. $2a, a^2+ab.$ 11. $6x^2, 4x^2+2x.$ 12. $(a+b)^2, a^2-b^2.$

13.* $2a+4b, 3a+6b.$ 14. $a^2-b^2, ab-b^2.$

15. $m^2-n^2, m^2-2mn+n^2.$ 16. $x^2+xy, xy+y^2, (x+y)^2.$

17. $mn+2n, m^2+3m+2.$ 18. $a^2-3a+2, a^2-5a+6.$

19. $x^2-9, x^2-7x+12, x^2-4x+3.$ 20. $y^2+2y-3, y^2+y-2.$

21. $a^2+2ab+b^2, 2a^2-2b^2.$ 22. $x^2-10x+25, 3x^2-75.$

23. $6ab+4b^2, 6a^2+4ab.$ 24. $a^3-2a^2+a, a^3+a^2-2a.$

25. If $a+b$ is a common factor of $a^2+mab+b^2$ and $a^2+nab+2b^2$, what are the values of m and n ?

26. The H.C.F of a^2b and ab^2 is ab. Find the greatest common measure of the numbers to which a^2b and ab^2 are equal when $a=2$ and $b=4$, and compare the result with the value of ab when $a=2$ and $b=4$.

71. **Algebraic Fractions.** A fraction has the same meaning in algebra as it has in arithmetic.

Thus, $\frac{3}{4}$ means 3 of the 4 equal parts of a unit, or the quotient of $3 \div 4$.

Similarly, $\frac{a}{b}$ means a of the b equal parts of a unit, or the quotient of $a \div b$.

The fraction $\frac{a}{b}$ is read " a divided by b " or " a over b."

72. **Changes in the Terms of a Fraction.** As in arithmetic, both terms of a fraction may be multiplied or divided by the same quantity (zero being excepted) without altering the value of the fraction.

Thus, [illegible] = etc.

Similarly, $\frac{a}{b}=\frac{ac}{bc}=\frac{acx}{bcx}=$ etc.,

and $\frac{a^2b^2}{a^2bc}=\frac{ab^2}{abc}=\frac{b^2}{bc}=\frac{b}{c}$.

73. Lowest Terms. A fraction is said to be in its lowest terms when its numerator and denominator have no common factor. If it is not in its lowest terms, it may be reduced by dividing both terms by all the common factors.

Examples.

1. $\frac{18}{42}=\frac{18\div6}{42\div6}=\frac{3}{7}$.
2. $\frac{15a^3b}{25ab^3}=\frac{15a^3b\div5ab}{25ab^3\div5ab}=\frac{3a^2}{5b^2}$.
3. $\frac{x^2-y^2}{x^2+2xy+y^2}=\frac{(x+y)(x-y)}{(x+y)(x+y)}=\frac{x-y}{x+y}$.
4. $\frac{x^2+5xy+4y^2}{x^2+3xy-4y^2}=\frac{(x+y)(x+4y)}{(x-y)(x+4y)}=\frac{x+y}{x-y}$.

The attention of the pupil is drawn to the fact that it is factors and not terms which are cancelled from the numerator and denominator.

Thus, in the fraction $\frac{7+2}{9+2}$ we cannot cancel the twos and say that the fraction is equal to $\frac{7}{9}$, for the value of the fraction is $\frac{9}{11}$, which does not equal $\frac{7}{9}$. But if the fraction is $\frac{7\times2}{9\times2}$ we can now cancel the twos and the resulting fraction is $\frac{7}{9}$.

Similarly, $\frac{ab}{ac}=\frac{b}{c}$ after cancelling, or dividing by the common factor a. But $\frac{a+b}{a+c}$ is not equal to $\frac{b}{c}$.

It is thus seen, *that no cancelling can be done until both terms of the fraction are expressed as products.*

EXERCISE 56 (6-21 Oral)

Fill in the blanks in the following:

1. $\frac{15}{20}=\frac{}{4}=\frac{30}{}=\frac{}{12}=\frac{}{4a}=\frac{6xy}{}.$

2. $\frac{ax}{bx}=\frac{}{b}=\frac{ac}{}=\frac{}{bm}=\frac{-5a}{}=\frac{3am}{}.$

3. $\frac{a}{b}=\frac{}{b(m+n)}=\frac{a(p+q)}{}=\frac{}{6b^2}.$

4. $\frac{6a^2x}{12a^2x^3}=\frac{a^2x}{}=\frac{a^2x}{}=\frac{}{2x^2}=\frac{a}{}.$

5. $\frac{a^2-b^2}{a^2-2ab+b^2}=\frac{(\quad)(\quad)}{(\quad)^2}=\frac{}{a-b}.$

Reduce to lowest terms:

6. $\frac{14}{21}.$
7. $\frac{3x}{6}.$
8. $\frac{15a}{25}.$
9. $\frac{3b}{6c}.$
10. $\frac{ab}{ac}.$
11. $\frac{4x^2}{6x}.$
12. $\frac{10m^2n}{15mn}.$
13. $\frac{35a^2b}{5ab}.$
14. $\frac{2a+4}{6}.$
15. $\frac{3a^2+6a}{12a}.$
16. $\frac{4x}{2x^2-8x}.$
17. $\frac{a(x-3)}{b(x-3)}.$
18. $\frac{x(x-1)}{(x-1)^2}.$
19. $\frac{(x-1)(x-2)}{(x-2)(x-3)}.$
20. $\frac{3a(x-y)}{6a^2}.$
21. $\frac{a(a-b)}{(a-b)^2}.$
22.* $\frac{x^2-1}{x^2-x}.$
23. $\frac{y^2-y}{y^2-2y+1}.$
24. $\frac{x^2-3x+2}{x^2-4x+3}.$
25. $\frac{m^2+7m+12}{m^2+4m}.$
26. $\frac{a^2-b^2}{a^2+5ab+4b^2}.$
27. $\frac{3x^2-3y^2}{3x^2+9xy+6y^2}.$
28. $\frac{a^3-a}{a^2-a}.$
29. $\frac{2a^4-2}{4a^2-4}.$
30. $\frac{x^5-x}{x^3-x}.$

74. Multiplication and Division of Fractions. The methods by which fractions are multiplied and divided in algebra are the same as in arithmetic.

EXAMPLES.

1. $\frac{3}{4} \times \frac{5}{7} = \frac{3 \times 5}{4 \times 7} = \frac{15}{28}$.

2. $\frac{10}{21} \times \frac{7}{3} \div \frac{4}{15} = \frac{10}{21} \times \frac{7}{3} \times \frac{15}{4} = \frac{25}{6} = 4\frac{1}{6}$.

3. $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d} = \frac{ac}{bd}$.

4. $\frac{ab}{xy} \times \frac{x^2y}{a^2} \div \frac{x^2}{a} = \frac{ab}{xy} \times \frac{x^2y}{a^2} \times \frac{a}{x^2} = \frac{b}{x}$.

5. $\frac{a^2+ab}{c^2+cd} \times \frac{cd+d^2}{ab+b^2} = \frac{a(a+b)}{c(c+d)} \times \frac{d(c+d)}{b(a+b)} = \frac{ad}{bc}$.

EXERCISE 56

Simplify:

1. $\frac{4}{5} \times \frac{5}{6} \times \frac{3}{4}$.
2. $\frac{2}{15} \times \frac{5}{7} \div \frac{3}{14}$.
3. $\frac{2a}{3b} \times \frac{9b}{4c}$.
4. $\frac{a}{b} \times \frac{b}{c} \times \frac{c}{a}$.
5. $\frac{ab}{xy} \times \frac{yz}{bc}$.
6. $\frac{2a}{3b} \div \frac{4a}{3b}$.
7. $\frac{a}{b} \div \frac{c}{d}$.
8. $\frac{25x^2}{14y^2} \div \frac{15x^2}{7y}$.
9. $\frac{6a^2}{5b} \div 12a$.
10.* $\frac{5x}{3x-3} \times \frac{x-1}{10}$.
11. $\frac{4a+6b}{5x} \times \frac{10x}{2a+3b}$.
12. $\frac{x^2+xy}{a^2+ab} \div \frac{xy+y^2}{ab+b^2}$
13. $\frac{x^2-1}{x^2-4} \times \frac{x^2-5x+6}{x^2-4x+3}$.
14. $\frac{a^2-3a+2}{a^2-5a+6} \times \frac{a^2-7a+12}{a^2-6a+5} \times \frac{a^2-4a+3}{a^2-5a+4}$
15. $\frac{a^2-b^2}{a^2-3ab+2b^2} \times \frac{a^2+2ab-8b^2}{a^2-2ab-3b^2} \div \frac{a^2+3ab-4b^2}{a^2-4ab+3b^2}$.

75. Lowest Common Multiple. A product is a multiple of any of its factors.

Thus, $3xy$ is a multiple of 3, of x, of y, of $3x$, of $3y$, of xy, and a^4 is a multiple of a, of a^2, of a^3.

When an expression is a multiple of two or more expressions it is a **common multiple** of those expressions.

Thus, $12a^2b^2$ is a common multiple of $2a^2$ and $3ab^2$.

The lowest common multiple (L.C.M.) of two or more expressions is the expression containing the *smallest* number of factors which is a multiple of each of the given expressions.

Ex. 1.—Find the L.C.M. of $6x^2y$, $9xy^2$ and $12xy^3$.

The numerical coefficient of the L.C.M. is evidently the L.C.M. of 6, 9 and 12 or 36.

The highest power of x in any of the given expressions is x^2 and of y is y^3, so that the L.C.M. must contain the factors x^2 and y^3.

$\therefore$ the L.C.M. $= 36 \times x^2 \times y^3 = 36x^2y^3$.

Ex. 2.—Find the L.C.M. of a^2-b^2 and $a^2-2ab+b^2$.

$$a^2-b^2=(a-b)(a+b).$$
$$a^2-2ab+b^2=(a-b)^2.$$
$$\therefore \text{ the L.C.M. } =(a-b)^2(a+b).$$

Why is $(a-b)(a+b)$ or $(a-b)^2(a+b)^2$ not the L.C.M. ?

EXERCISE 57 (1-9, Oral)

Find the L.C.M. of :

1. 3, 4, 5.
2. 10, 15, 20.
3. $2a$, $4a$, $6a$.
4. a, ab, a^2.
5. x^2, xy, y^2.
6. $2ab$, $3ac$, $6bc$.
7. $10a^2$, $15a^2$, $5a$.
8. $3a^2$, $2a^2$, $4a$.
9. $6a^2b$, $4ab^2$.
10.* a^2, a^2+a.
11. $3x$, $3x^2+6x$.
12. $ab+ac$, b^2+bc.
13. $2x+2$, x^2-1.
14. x^2+xy, $(x+y)^2$.
15. x^2-1, x^2-3x+2
16. a^2-ab, $ab-b^2$.
17. a^2-b^2, $a^2-2ab+b^2$.
18. x^2-x, x^3-x.
19. $2x$, $4x+4$, $2x^2-2$.
20. y^2-3y+2, y^2-y-2, y^2-1.
21. Show that the product of x^2+x-2 and x^2-x-6 is equal to the product of their H.C.F. and L.C.M.

76. Addition and Subtraction of Fractions. If we wish to add or subtract fractions we must reduce them to a common denominator. As in arithmetic, the lowest common denominator is the L.C.M. of the denominators.

EXAMPLES.

1. $\frac{3}{4}+\frac{5}{6}-\frac{2}{3}=\frac{9}{12}+\frac{10}{12}-\frac{8}{12}=\frac{9+10-8}{12}=\frac{11}{12}.$

2. $\frac{a}{b}+\frac{a}{c}=\frac{ac}{bc}+\frac{ab}{bc}=\frac{ac+ab}{bc}.$

3. $\frac{3}{a^2}-\frac{4}{b^2}+\frac{5}{ab}=\frac{3b^2}{a^2b^2}-\frac{4a^2}{a^2b^2}+\frac{5ab}{a^2b^2}=\frac{3b^2-4a^2+5ab}{a^2b^2}.$

4. $\frac{a}{x^2+xy}-\frac{b}{xy+y^2}=\frac{a}{x(x+y)}-\frac{b}{y(x+y)}=\frac{ay-bx}{xy(x+y)}.$

5. $\frac{2x}{x^2-4}-\frac{2}{x+2}=\frac{2x}{(x+2)(x+2)}-\frac{2}{x+2}=\frac{2x-2(x-2)}{(x+2)(x-2)}$

$$=\frac{2x-2x+4}{(x+2)(x-2)}=\frac{4}{(x+2)(x-2)}.$$

EXERCISE 38 (1-8, Oral)

Reduce to fractions with the lowest common denominator:

1. $\frac{2}{3}, \frac{5}{9}.$ 2. $\frac{3}{4}, \frac{a}{4b}.$ 3. $\frac{1}{a}, \frac{1}{a^2}.$ 4. $\frac{2}{3x}, \frac{3}{2x}.$

5. $\frac{3a}{4}, \frac{4a}{8}.$ 6. $\frac{m}{n}, \frac{n}{m}.$ 7. $\frac{1}{a}, \frac{b}{c}.$ 8. $\frac{2}{a}, \frac{3b}{a^2}.$

9. $\frac{2}{3a}, \frac{5}{4a^2}, \frac{3}{2a^3}.$ 10. $\frac{a}{b}, \frac{b}{c}, \frac{c}{a}.$ 11. $\frac{2x^2}{b}, \frac{3bx}{2}, \frac{4}{3bc}.$

12. $\frac{x}{3y^2}, \frac{4}{3xy}, \frac{b}{2x}.$ 13. $\frac{x}{3ab}, \frac{y}{2bc}, \frac{z}{4ac}.$ 14. $\frac{a+1}{a}, \frac{a-1}{2a}, \frac{a+2}{3a}.$

Perform the operations indicated:

15. $\frac{a}{3}+\frac{a}{4}.$ 16. $\frac{a+b}{2}+\frac{a-b}{3}.$ 17. $\frac{a+4}{3}+\frac{5-a}{5}.$

18. $\frac{a-x}{3}-\frac{a}{5}.$ 19. $\frac{3}{x-1}+\frac{2}{x+1}.$ 20. $\frac{a+b}{2}+\frac{b+c}{3}+\frac{a+c}{4}.$

21. $\frac{1}{1-x}+\frac{1}{1+x}.$ 22.* $\frac{x+3}{3}-\frac{4-x}{6}.$ 23. $\frac{x+y}{4}+\frac{x-y}{2}-\frac{x+y}{8}.$

24. $\frac{x-y}{2}-\frac{x-y}{3}+\frac{x+y}{4}.$ 25. $\frac{x}{x-y}-\frac{y}{x+y}.$

26. $\dfrac{4}{a+4}+\dfrac{4}{a-4}-\dfrac{8a}{a^2-16}$.

27. $\dfrac{a+x}{a}+\dfrac{a-x}{x}+\dfrac{a^2-x^2}{ax}$.

28. $\dfrac{x}{3x+6}-\dfrac{x}{2x+4}$.

29. $\dfrac{2}{a^2-ab}-\dfrac{3}{ab-b^2}$.

30. $\dfrac{1}{a^2+3a+2}+\dfrac{2}{a^2+5a+6}+\dfrac{3}{a^2+4a+3}$. (Check when $a=1$.)

31. $\dfrac{2}{a^2-1}+\dfrac{1}{a^2+3a+2}-\dfrac{1}{a^2+a-2}$.

77. **Mixed Expressions.** An expression which is partly integral and partly fractional is called a **mixed expression.** A mixed expression in algebra corresponds to a mixed number in arithmetic.

Thus, $3\frac{2}{5}$ is a mixed number and $a+\dfrac{b}{c}$ is a mixed expression.

Note that in a mixed number the sign of addition is omitted and $3\frac{2}{5}$ means $3+\frac{2}{5}$. But in algebra the sign must be inserted, as $a\dfrac{b}{c}$ would mean $a\times\dfrac{b}{c}$ and not $a+\dfrac{b}{c}$.

78. **Reduction of a Mixed Expression to a Fraction.** Since every integral quantity may be written as a fraction whose denominator is unity, it follows that the reduction of a mixed expression to a complete fraction is a problem in addition or subtraction.

EXAMPLES.

1. $3\frac{2}{5}=3+\dfrac{2}{5}=\dfrac{15+2}{5}=\dfrac{17}{5}$.

2. $a+\dfrac{b}{c}=\dfrac{ac}{c}+\dfrac{b}{c}=\dfrac{ac+b}{c}$.

3. $5-\dfrac{x}{y}=\dfrac{5y}{y}-\dfrac{x}{y}=\dfrac{5y-x}{y}$.

4. $a-\dfrac{ac}{b+c}=\dfrac{a(b+c)}{b+c}-\dfrac{ac}{b+c}=\dfrac{ab+ac-ac}{b+c}=\dfrac{ab}{b+c}$.

79. Reduction of a Fraction to a Mixed Expression. To separate $\frac{ab+bc}{a}$ into two fractions we merely reverse the operation of addition.

Thus, $$\frac{ab+bc}{a}=\frac{ab}{a}+\frac{bc}{a}=b+\frac{bc}{a},$$

and $$\frac{ab-bd-c}{b}=\frac{ab}{b}-\frac{bd}{b}-\frac{c}{b}=a-d-\frac{c}{b}.$$

EXERCISE 59 (1-9, Oral)

Reduce to complete fractions:

1. $2+$[illegible].
2. $1+\frac{a}{2}$.
3. $3+\frac{x}{y}$.
4. $a+\frac{m}{4}$.
5. $x-\frac{y}{3}$.
6. $a-\frac{b}{c}$.
7. $x-\frac{m^2}{n}$.
8. $2x+\frac{3y}{x}$.
9. $ab-\frac{bd}{c}$.
10. $a-\frac{ab}{b+c}$.
11. $x+\frac{xy}{x-y}$.
12. $2a+\frac{2ab}{a-b}$.
13. $x+1+\frac{2}{3x}$.
14. $a-b-\frac{a+b}{2}$.
15. $x-y+\frac{x^2+y^2}{x+y}$

Separate into fractions in their lowest terms:

16. $\frac{6a+2b}{4}$.
17. $\frac{ax+bx}{ab}$.
18. $\frac{5x-8y}{10a}$.
19. $\frac{6a^2-3b^2}{3ab}$.
20. $\frac{a+7b-3c}{21a}$.
21. $\frac{9y-14x}{6xy}$.
22. $\frac{3mn-4n}{2n}$.
23. $\frac{6abc-9bc+c^2}{3bc}$.
24. $\frac{(a-b)^2+x}{a-b}$.

EXERCISE 60 (Review of Chapter IX)

1. Define highest common factor and lowest common multiple.
2.* Find the H.C.F. and L.C.M. of $3x-6$, $4x-8$, $5x-10$.

3. Find the H.C.F. and L.C.M. of x^2+xy, $xy+y^2$ and x^2y+xy^2.

4. Find the H.C.F. and L.C.M. of $x^2-7x+10$ and x^2+2x-8. Show that the product of these expressions is equal to the product of their H.C.F. and L.C.M.

5. Reduce to lowest terms:

$$\frac{a^2+ab}{a^2},\ \frac{x^2}{x^2-xy},\ \frac{6a^2-9ab}{8ab-12b^2},\ \frac{abx-bx^2}{acx-cx^2}.$$

6. Multiply $\frac{6ax}{5by}$, $\frac{4cy}{3ad}$, $\frac{5bd}{2cx}$.

7. Simplify $\frac{2c^2}{5ab}\times\frac{3bd}{4ac}\div\frac{6cd}{5a^2}$.

8. Reduce to lowest terms:

$$\frac{x^2-2x}{x^2-5x+6},\ \frac{x^2+4x+4}{x^2+5x+6},\ \frac{2x^2-18}{3x^2+3x-18},\ \frac{a^2-b^2}{a^2-2ab+b^2}.$$

9. Simplify $\frac{x-y}{x^2+2xy+y^2}\times\frac{x+y}{x^2-2xy+y^2}\times\frac{2ax}{y}\times\frac{xy-y^2}{x^2+xy}$.

10. Divide $\frac{a-x}{a^2+2ax}$ by $\frac{a^2-ax}{a^2-4x^2}$ and $\frac{x+2}{x+1}$ by $\frac{x^2-4}{x^2-1}$.

11. Simplify $\frac{x-3}{3}+\frac{x+4}{4}$ and $\frac{2x-1}{3}+\frac{8-4x}{6}$.

12. Find the sum of $\frac{x-y}{xy}$, $\frac{y-z}{yz}$, $\frac{z-x}{zx}$.

13. From the sum of $\frac{3b+4a}{2ab}$ and $\frac{b-6c}{2bc}$ subtract $\frac{a+6c}{4ac}$.

14. Simplify $\frac{x-3}{3x}-\frac{x-5}{5x}$ and $\frac{3x-y}{xy}-\frac{3z-2y}{yz}$.

15. Express $\frac{a^2-b^2}{a^2b^2}$ as the difference of two fractions in their lowest terms. Do the same with $\frac{b^2-c^2}{b^2c^2}$ and $\frac{c^2-a^2}{c^2a^2}$ and find the sum of the three results.

16. By how much does $\frac{y-4}{4y}$ exceed $\frac{y-5}{5y}$?

17. Find the sum of $\frac{1}{a+b}$, $\frac{1}{a-b}$ and $\frac{2a}{a^2-b^2}$.

18. By what must $\frac{x-3}{x-4}$ be multiplied to give $\frac{x^2-5x+6}{x^2-7x+12}$ as the product?

19. Find the quotient when $\frac{x+4}{x-1}$ is divided by $\frac{x^2+x-12}{x^2-4x+3}$.

20. Solve $(a-b)x=(a^2-b^2)(a+b)$.

21. Find the difference between

$$\frac{a}{a-x}+\frac{b}{b-x}+\frac{c}{c-x} \quad\text{and}\quad \frac{x}{a-x}+\frac{x}{b-x}+\frac{x}{c-x}$$

by first subtracting $\frac{x}{a-x}$ from $\frac{a}{a-x}$, etc.

22. Find the missing term in the following identity:

$$\frac{x^2-5x+6}{x^2-3x-4}\times\frac{x^2+5x\ldots}{x^2-9}=\frac{x^2+2x-8}{x^2-x-12}.$$

CHAPTER X

REVIEW OF THE SIMPLE RULES

80. In this Chapter will be found such exercises as will furnish a review of the elementary rules. In it is also included matter which it was not thought advisable to present to beginners in the subject of algebra.

81. Use of Brackets. We have already seen that

(1) $a+(b+c)=a+b+c$,

(2) $a+(b-c)=a+b-c$,

(3) $a-(b+c)=a-b-c$,

(4) $a-(b-c)=a-b+c$.

That is, when brackets are preceded by the negative sign, as in (3) and (4), the brackets may be removed if *the signs of all terms within the brackets be changed*; but when they are preceded by the positive sign, as in (1) and (2), *no change is made in the signs when the brackets are removed.*

In (3) the sign of b in $a-(b+c)$ is positive as the expression might be written $a-(+b+c)$. When the brackets are removed we follow the rule and change $+b$ into $-b$.

Sometimes we find more than one pair of brackets in the same expression.

Ex. 1.—Simplify $a-(3a-2b)+(5a-4b)$.

Following the rule, this expression becomes

$$a-3a+2b+5a-4b \text{ or } 3a-2b.$$

When one pair of brackets appears within another, it is better to remove the brackets singly, and the pupil is advised to remove the inner brackets first.

Ex. 2.—Simplify $4x-\{2x-(3+x)\}$.

Removing the inner brackets, we get

$$4x-\{2x-3-x\}.$$

Removing the remaining brackets, we get

$$4x-2x+3+x \text{ or } 3x+3.$$

Ex. 3.—Simplify $3a-[a+b-\{a-b-c-(a+b-c)\}]$.

$$\begin{aligned}\text{The expression} &= 3a-[a+b-\{a-b-c-a-b+c\}],\\ &= 3a-[a+b-a+b+c+a+b-c],\\ &= 3a-a-b+a-b-c-a-b+c,\\ &= 2a-3b.\end{aligned}$$

After removing the first pair of brackets, the quantity

$$a-b-c-a-b+c$$

might have been changed into the simple form $-2b$. Work the problem again, simplifying at each step.

When brackets are used to indicate multiplication, the multiplication must be performed if the brackets are removed.

Ex. 4.—Simplify $4x-3(x-2y)+\overline{2x-4y}$.

$$\begin{aligned}\text{The expression} &= 4x-(3x-6y)+\overline{2x-4y},\\ &= 4x-3x+6y+2x-4y,\\ &= 3x+2y.\end{aligned}$$

NOTE.—When the pupil has had some practice he should be able to remove the brackets and perform the multiplication in a single step.

EXERCISE 61 (1-9, Oral)

Remove the brackets from:

1. $(a-b)+(c-d)$.
2. $(a-b)-(c-d)$.
3. $-(a-b)+(c-d)$.
4. $-(a-b)-(c-d)$.
5. $a-(b-c)-(d-e)$.
6. $a-(-b)-(-c)$.
7. $a+\{b+(c-d)\}$.
8. $a+\{b-(c-d)\}$.
9. $a-\{b+(c-d)\}$.
10. $a-\{b-(c-d)\}$.
11. $a-\{-b-(c-d)\}$.
12. $-[a-\{b-(c-d)\}]$.

Simplify:

13. $4a-2b-(2a-2b)$.

14. $2(7x-3y)-3(2x-3y)$.

15. $3(a-b+c)-2(a+b-c)$.

16. $2a-\{3a+2(a-2b)\}$.

17. $3(a+b-c)-2(a-b+c)+5(b-c+a)$.

18.* $15x-\{4-[3-5x-(3x-7)]\}$.

19. Add $3(a+b)-5(p+q)$, $-2(a+b)+(p+q)$ and $4(p+q)$.

20. Add $1+\overline{x-y}$, $1-\overline{x-y}$ and $1-\overline{x+y}$.

21. Add $3x-2(y-z)$, $3y-2(z-x)$, $3z-2(x-y)$.

Remove the brackets and express in descending powers of x:

22. $3(5x-3+2x^2)-2(x^2-5+3x)-3(4-5x-6x^2)$.

23. $2x(3x-2)-5(x-3)+6x(x-1)-2(x^2-5x)$.

24. $\frac{1}{2}(4x-3)-\frac{1}{4}(6-x^2)+\frac{3}{4}(x^2+8x-12)$.

Solve for x and verify:

25. $4(x-3)-7(x-4)=6-x$.

26. $5x-[8x-3\{16-6x-(4-5x)\}]=6$.

27. $3(2x-7)-(x-14)-2(5x+17)=6(5-8x)+21x+149$.

28. $\frac{1}{3}(27-2x)=\frac{9}{2}-\frac{1}{10}(7x-54)$.

29. Simplify $a-[5b-\{a-(3c-3b)+2c-(a-2b-c)\}]$.

30. Simplify $\dfrac{3(a-b+c)+2(b-c+a)-(c-a+b)}{5(a-2b+c)-2(b-3c+2a)-(11c-2a-11b)}$.

31. Solve $(7\frac{1}{3}x-2\frac{1}{2})-[4\frac{1}{3}-\frac{1}{2}(3\frac{1}{3}-5x)]=18\frac{1}{3}$.

82. Insertion of Quantities in Brackets. The trinomial $a-b+c$ may be changed into a binomial by combining two of its terms into a single term. This may be done in a number of ways.

Thus,
$$a-b+c=(a-b)+c=(a+c)-b$$
$$=a-(b-c)=a+(c-b).$$

Remove the brackets mentally and see that each of these is equal to $a-b+c$.

Ex. 1.—Express $a-b+c-d$ as a binomial.

As we have seen, this may be done in many ways as $a-(b-c+d)$, $(a-b)+(c-d)$, $(a+c)-(b+d)$, $(a-d)-(b-c)$, $c-(b+d-a)$.

Note.—The pupil must exercise particular care when dealing with brackets which are preceded by the negative sign. The signs of all terms inserted in such brackets must, of course, be changed. He should, in every case, remove the brackets mentally to test the accuracy of the work.

Ex. 2.—Express $a+b-c$ as a binomial by combining the last two terms within brackets, preceded by the negative sign.

$$\begin{aligned} a+b-c &= a-(-b+c), \\ &= a-(c-b). \end{aligned}$$

Either form is correct, but it is usual to make the first term within the brackets positive, so that the second form is preferable.

83. Collecting Coefficients. Brackets are frequently used for the purpose of **collecting the coefficients** of particular letters in an expression.

Thus, $\quad ax+by-cx-dy=x(a-c)+y(b-d)$,

and $\quad mx-ny+px+qy=x(m+p)-y(n-q)$,

$$=x(m+p)+y(q-n).$$

Verify these by removing the brackets.

EXERCISE 62

1. Express $3a-2b+4c$ as a binomial in three different ways and verify in each case.

2. Express $p-q-r+s$ as a trinomial in four different ways and verify.

3. Express $x-y-z-k$ as a binomial in four different ways and verify.

Collect the coefficients of x and y:

4. $ax-by-cx-dy$.

5. $mx-ny-px+qy-ax+by$.

6. $a(x-y)+b(2y-3x)+c(5x+2y)$.

7. $x(a-b)+y(b-c)-d(x+y)$.

8. $2ax-3by-10x-5y+6bx-4ay$.

9. $(a-3)y-(2-b)x+4y+2ax-(3x+by)$.

10. Enclose $a-b-c-d-e+f$ in alphabetical order in brackets, with two terms in each; with three terms in each.

Arrange in descending powers of x:

11. $a(x^2+4-3x)-b(3x-5x^3)-c(1-4x)$.

12. $ax^3-bx+c-(2px^2-3qx+r)-(7dx^3+3cx+f)$.

84. Multiplication with Detached Coefficients. The method of multiplying two binomials has already been shown in Chapter V. The same method is followed when the factors are not binomials.

Ex. 1.—Multiply x^2-3x+4 by $x-2$.

$$\begin{array}{r}
(1)\\
x^2-3x+4\\
x-2\\
\hline
x^3-3x^2+\ \ 4x\qquad\\
-2x^2+\ \ 6x-8\\
\hline
x^3-5x^2+10x-8
\end{array}
\qquad
\begin{array}{r}
(2)\\
1-3+4\\
1-2\\
\hline
1-3+\ \ 4\qquad\\
-2+\ \ 6-8\\
\hline
1-5+10-8
\end{array}
\qquad
\begin{array}{r}
(3)\\
\text{Check}\\
x=1\\
\hline
+2\\
-1\\
\hline
-2
\end{array}$$

The second method is called multiplication with **detached coefficients.** The processes in the two methods are the same, with the exception that the letters are omitted in the second method and the coefficients only are used.

When the second method is used the first coefficient in the product must be the coefficient of the product of x^2 and x, that is, of x^3. The next must be the coefficient of x^2 and the next of x, as the product will evidently be in descending powers of x, as both factors multiplied are so written.

In (3) the check is shown as explained in art. 42.

Ex. 2.—Multiply $3x^3-7x+2$ by x^2-2x+3.

Here the term containing x^2 in the first expression is missing and a zero is supplied in order to bring coefficients of like powers of x under each other in the partial products.

The first term in the product is $3x^5$.

Write down the complete product and check the work.

$$\begin{array}{r}
3+0-7+\ \ 2\qquad\qquad\qquad\\
1-2+3\qquad\qquad\qquad\qquad\\
\hline
3+0-7+\ \ 2\qquad\qquad\qquad\\
-6-0+14-\ \ 4\qquad\qquad\\
+9+\ \ 0-21+6\\
\hline
3-6+2+16-25+6
\end{array}$$

Ex. 3.—Find the coefficient of x^3 in the product of

$$5x^3-6x^2+3x-2 \text{ and } x^3-2x^2-3x+4.$$

Here the complete product is not required, but only the term which contains x^3.

The partial products which will contain x^3 are evidently those which we obtain by multiplying -2 by x^3, $3x$ by $-2x^2$, $-6x^2$ by $-3x$ and $5x^3$ by 4.

$\therefore$ the coefficient of $x^3 = -2-6+18+20=30.$

Ex. 4.—Multiply ax^2+bx+c by $mx-n$.

Here the multiplication is performed in the usual way. In adding the partial products, the coefficients of the powers of x are collected.

$$\begin{array}{llll}
ax^2 & +bx & +c & \\
mx & -n & & \\
\hline
amx^3+bmx^2 & +cmx & & \\
\quad -anx^2 & -bnx & -cn & \\
\hline
\multicolumn{4}{l}{amx^3+(bm-an)x^2+(cm-bn)x-cn.}
\end{array}$$

EXERCISE 63

Multiply and check:

1. x^2-3x+2, $x-2$.
2. $2x^2-5x-3$, $3x-2$.
3. x^2-x+1, $x+1$.
4. a^2+ab+b^2, $a-b$.
5. x^2-x+1, x^2+x+1.
6. a^3-5a^2-2, a^2+a-1
7. $3x^2-2x-5$, x^2+x-3.
8. $2a^2-5ab+3b^2$, $2a^2+5ab-3b^2$
9. $a+b-c$, $a-b+c$.
10. x^3+2x^2+4x+8, x^2-4x+4.
11. b^2-b+1, b^2+b+1, b^4-b^2+1.
12. $x^2-xy+y^2+x+y+1$, $x+y-1$.

Use detached coefficients to multiply; check the results:

13. $3x^3-4x^2+7x-3$ by x^2-2x-1.
14. $5a^4-6a^3-2a^2-a+2$ by $2a^2-3a+2$.
15. $4x^3-5x-2$ by $4x^2-3x-1$.
16. $(x^2-x-2)(2x^2-x-1)(3x-2)$.

Simplify:

17.* $(x-1)(x-2)+(x-2)(x-3)+(x-3)(x-1)$.

18. $(a+x)(b-c)+(b+x)(c-a)+(c+x)(a-b)$.

19. $(a+b)(c+d)-(a-b)(c-d)$.

20. $(a+b-c)(a-b)+(b+c-a)(b-c)+(c+a-b)(c-a)$.

21. $(x+1)(x+2)(x+3)-(x-1)(x-2)(x-3)$.

Find the product of:

22. $(1-x)(1+x)(1+x^2)(1+x^4)$. 23. $(x-1)(x-2)(x-3)(x-4)$.

24. $(x-1)(x-3)(x+1)(x+3)$. 25. $(a-1)(a^2+a+1)(a^2+1)$.

Find the coefficient of x^3 in the product of:

26. $3x^2-5x+11$ and $5x^2+6x-4$.

27. x^3+4x^2-5x+2 and x^2-2x-3.

28. $3x^2-12x+15$ and $2x^2-7x-38$.

29. Multiply $1+x+x^2+x^3$ by $1+2x+3x^2+4x^3$, retaining no powers higher than the third.

30. Add together $(x-1)(x+2)$, $(x+2)(x-3)$, $(x+3)(x+4)(x-1)$, $(x+4)(x^2-2x+3)$ and $7-x^3+3x$.

Check by putting $x=2$.

31. Multiply $7x^3-5x^2y-xy^2+6y^3$ by $4x^2+3xy-2y^2$.

32. Show that the expression $x(x+1)(x+2)(x+3)+1$ is equal to $(x^2+3x+1)^2$.

33. Find the first four terms only in the product of:

$$2+3x+4x^2+5x^3 \text{ and } 1-2x+3x^2-4x^3.$$

34. Find the coefficient of x^4 in the product of:

$$1+4x+7x^2+10x^3+13x^4 \text{ and } 1+5x+9x^2+13x^3+17x^4.$$

35. Solve and verify:

$$(x-2)(x-4)(x-6)(x-10)=(x-1)(x-5)(x-7)(x-9).$$

36. Multiply ax^2+bx+c by bx^2-cx+d. Collect the coefficients of x and write in descending powers.

37. Multiply px^2-qx+r by $px+q$, and $(a-1)x^2+ax-1$ by $ax+1$.

38. Simplify $(ay^2-by+c)(cy+b)+(ay^2+by-c)(ay-b)$.

39. Subtract the product of $x^2+x(p+1)-1$ and $x-2p$ from the product of $x^2-x(p-1)+2$ and $x+p$.

40. Point out two obvious errors in each of the following statements:

(i) $ab(a+b)(a^2+b^2)=a^4b+a^3b+a^2b^3-ab^4$.

(ii) $(2x+3y)^3=6x^3+36xy-54xy^2+27y^3$.

(iii) $x^3-6x^2y-3xy^2+2y^3=(x-2y)(x^2-4x+y^2)$.

41. Use the formula $(a+1)(b+1)=ab+(a+b)+1$ to find the product of 2146 and 3526, being given that the product of 2145 and 3525 is 7,561,125.

85. Division by a Compound Quantity. The method of dividing by a monomial has already been shown in Chapter V.

The method of dividing by a quantity containing two or more terms is in many ways similar to long division in arithmetic.

Divide 672 by 32.

(1)

$$\begin{array}{l} 32)672(21 \\ \underline{64} \\ 32 \\ \underline{32} \end{array}$$

(2)

$$\begin{array}{l} 3\,.\,10+2)6\,.\,10^2+7\,.\,10+2(2\,.\,10+1 \\ \underline{6\,.\,10^2+4\,.\,10} \\ 3\,.\,10+2 \\ \underline{3\,.\,10+2} \end{array}$$

In (2) the divisor is expressed in the equivalent form $3\,.\,10+2$ and the dividend $6\,.\,10^2+7\,.\,10+2$.

If we substitute x for 10 the problem would be:

Divide $6x^2+7x+2$ by $3x+2$.

$$\begin{array}{l} 3x+2)6x^2+7x+2(2x+1 \\ \underline{6x^2+4x} \\ 3x+2 \\ \underline{3x+2} \end{array}$$

The method here is so similar to the method in arithmetic, that little explanation is necessary. The first term in the quotient is obtained by dividing $3x$ into $6x^2$. The product of $3x+2$ by $2x$ is then subtracted from the dividend and the remainder is $3x+2$. The last term of the quotient is obtained by dividing the first term of the remainder ($3x$) by the first term of the divisor ($3x$).

In more complicated examples the method is precisely the same as here. The division is continued until there is no remainder, or until a remainder is found which is of lower degree than the divisor.

86. Verifying Division. The work may be verified as in arithmetic, by multiplication. It is simpler, however, to test by substituting a particular number for each letter involved.

Thus, in the preceding problem if we let $x=1$, the divisor is 5, the dividend is 15 and the quotient is 3, which shows that the result is very likely correct.

If on substituting particular values for the letters involved, the divisor becomes zero, other values should be selected.

87. Division with Detached Coefficients. As in multiplication the method of detached coefficients may be used.

Ex.—Divide $14x^4-x^3-29x^2+12$ by $7x^2+3x-6$.

$$\begin{array}{lr}
 & 7+3-6)14-1-29+0+12(2-1-2 \\
\text{Check} & 14+6-12 \\
x=1 & \overline{-7-17+0} \\
4)-4(-1 & -7-\ \ 3+6 \\
 & \overline{-14-6+12} \\
 & -14-6+12 \\
 & \overline{}
\end{array}$$

Here the first term in the quotient is $2x^2$, since $14x^4\div 7x^2=2x^2$. The complete quotient is $2x^2-x-2$.

Divide also by the usual method.

EXERCISE 64 (1-6, Oral)

State the quotients in the following divisions:

1. $\dfrac{x^2+3x+2}{x+1}$. 2. $\dfrac{a^2-3a+2}{a-1}$. 3. $\dfrac{a^2-b^2}{a-b}$.

4. $\dfrac{x^2-4}{x+2}$. 5. $\dfrac{a^2+2ab+b^2}{a+b}$. 6. $\dfrac{3x^2-5x+2}{3x-2}$.

Divide and verify:

7. $6x^2+x-15$ by $2x-3$. 8. $6x^2+xy-12y^2$ by $3x-4y$.

9. $5x^2-31xy+6y^2$ by $x-6y$. 10. $9a^2+6ab-35b^2$ by $3a+7b$.

11. $7x^3+96x^2-28x$ by $7x-2$.
12. $100x^3-13x^2-3x$ by $25x+3$.
13. $3+7x-6x^2$ by $3-2x$.
14. $6a^2+35-31a$ by $2a-7$.
15. $x^3+13x^2+54x+72$ by $x+6$.
16. $2a^3+7a^2+5a+100$ by $a+5$.
17. $x^3+3x^2y+3xy^2+y^3$ by $x+y$.
18. $-x^3+3x^2y-3xy^2+y^3$ by $x-y$
19. $16m^3-46m^2+39m-9$ by $8m-3$.
20. $6x^3-29x^2y+18xy^2+35y^3$ by $2x-7y$.
21. $a^4+a^3+4a^2+3a+9$ by a^2-a+3.
22. $x^4-x^3-6x^2+15x-9$ by x^2+2x-3.
23. $5x^4-4x^3+3x^2+22x+55$ by $5x^2+11x+11$
24. $2x^3-8x+x^4+12-7x^2$ by x^2+2-3x.
25. $30-12x^2+x^4-x$ by $x-5+x^2$.

Use detached coefficients to divide:

26. x^3-3x^2+3x-1 by x^2-2x+1.
27. $6x^4-x^3-11x^2-10x-2$ by $2x^2-3x-1$.
28. $a^5-5a^3+7a^2+6a+1$ by a^2+3a+1.
29. $4x^2+9+x^4+3x+x^3$ by $2x+3+x^2$.
30.* $(x^2-x-2)(2x^2+x-1)$ by $2x^2-5x+2$.
31. $10x^3+17x^4-2x^5-11x^2-x+1$ by $2x^2+x-1$.
32. Divide a^3-1 by $a-1$ and a^3+1 by $a+1$.
33. Simplify $\dfrac{x^3+y^3}{x+y}-\dfrac{x^3-y^3}{x-y}$.
34. Simplify $\dfrac{a^3-1}{a^2+a+1}+\dfrac{a^3+1}{a^2-a+1}$.
35. Solve $\dfrac{6x^2+x-2}{2x-1}-\dfrac{3x^2+8x-3}{3x-1}=11$.
36. If $x(3a^2-a+1)+2=3a^3-7a^2+3a$, find x.

37. The dividend is $a^4+6a^3+6a^2-9a+2$, the quotient is a^2+3a-1. Find the divisor.

38. Divide $x^8+x^4y^4+y^8$ by $x^4-x^2y^2+y^4$ and divide the quotient by x^2-xy+y^2.

39. Simplify $\dfrac{abx^2-x(ac+b^2)+bc}{bx-c}+\dfrac{acx^2+x(ad+bc)+bd}{cx+d}$.

Without removing the brackets divide:

40. $ax^2+(b+ac)x+bc$ by $ax+b$.

41. $x^2+(2p-1)x+p(p-1)$ by $x+p$.

42. $a^2x^2-2abx+b^2-c^2$ by $ax-(b-c)$.

43. $a^2y^3+(2a^2+a)y^2+(a^2+2a)y+(a+1)$ by ay^2+ay+1.

88. Inexact Division. As in arithmetic, the divisor may not divide evenly into the dividend, and so there may be a remainder.

Thus, $34\div 5$ gives a quotient 6 and a remainder 4,

$$\therefore\ \tfrac{34}{5}=6+\tfrac{4}{5}\ \text{or}\ 34=6\,.\,5+4.$$

Similarly, when a^2+3a+5 is divided by $a+1$, the quotient is $a+2$ and the remainder is 3.

$$\therefore\ \frac{a^2+3a+5}{a+1}=a+2+\frac{3}{a+1},$$

or

$$a^2+3a+5=(a+1)(a+2)+3.$$

That is, **dividend = divisor × quotient + remainder.**

Ex.—Express $\dfrac{1+x^4}{1-x}$ as a mixed expression.

Here the quotient is

$$1+x+x^2+x^3$$

and the remainder is $2x^4$.

$$\begin{array}{l}
1-x)1\qquad\qquad\qquad +x^4(1+x+x^2+x^3\\
\quad\ \underline{1-x}\\
\qquad +x\\
\qquad \underline{+x-x^2}\\
\qquad\qquad +x^2\\
\qquad\qquad \underline{+x^2-x^3}\\
\qquad\qquad\qquad +x^3+x^4\\
\qquad\qquad\qquad \underline{+x^3-x^4}\\
\qquad\qquad\qquad\qquad 2x^4
\end{array}$$

$$\therefore\ \frac{1+x^4}{1-x}=1+x+x^2+x^3+\frac{2x^4}{1-x}.$$

Divide $1-x^5$ by $1+x$ and show that

$$\frac{1-x^5}{1+x}=1-x+x^2-x^3+x^4-\frac{2x^5}{1+x}.$$

In such cases the division may, of course, be continued to any number of terms.

EXERCISE 65

Find the remainder on dividing:

1.* $x^2-10x+25$ by $x-7$.
2. $a^2+20a+70$ by $a+5$.
3. $x^3-4x^2+5x+20$ by $x-1$.
4. y^3-7y^2+8y-1 by y^2-y+1.
5. x^3+y^3 by $x-y$.
6. x^3-y^3 by $x+y$.

Express as mixed quantities:

7. $\frac{x+2}{x+1}$. 8. $\frac{a+2b}{a-b}$. 9. $\frac{2a-3b}{a+b}$. 10. $\frac{5x^2+7x-3}{x+2}$.

Find four terms in the quotient of:

11. $1\div(1-x)$. 12. $1\div(1+x)$. 13. $\frac{1+x}{1-x}$. 14. $\frac{1+a+2a^2}{1-a+a^2}$.

15. When the dividend is a^2-3a+7, the quotient is a and the remainder is 7. Find the divisor.

16. Divide x^2-5x+a by $x-2$ and determine for what value of a the division will be exact.

17. If $x^2-mx+12$ is divisible by $x-3$, what must the quotient be and what is the value of m?

18. By division show that

$$\frac{a^2+b^2}{a-b}=a+b+\frac{2b^2}{a-b};\quad \frac{a^2+b^2}{a+b}=a-b+\frac{2b^2}{a+b}.$$

EXERCISE 66 (Review of Chapter X)

1. Add $x^3-2ax^2+a^2x+a^3$, $3x^3+3ax^2$, $2a^3-ax^2-x^3$.
2. Add $\frac{1}{2}a-\frac{2}{3}b$, $\frac{2}{3}a-\frac{1}{4}b$, $\frac{1}{3}a+\frac{2}{3}b$.
3. Subtract $8a+3b-5c$ from $11a-2b+5c-3d$.
4. Subtract $-3a+4b-c$ from zero.
5. Subtract $\frac{1}{2}a-\frac{2}{3}b+\frac{1}{4}c$ from $\frac{1}{3}a+\frac{2}{3}b-\frac{3}{4}c$.
6. How much must be added to $\frac{1}{2}x-\frac{1}{3}y+\frac{1}{4}z$ to produce $x-y+z$?
7. Subtract the sum of $3a+2b$, $2b-3c$ and $3c-a$ from the sum of $a-b$, $b-c$ and $c-a$.
8. Simplify $a-(3b-4c)-(b+c-a)-2(a-c)$.

9. Subtract $4x^3-3x^2-x+2$ from $7x^3-6x^2+2x-1$ and check by substituting 2 for x.

10. Find the value of $a^3+b^3+c^3-3abc$ when $a=2$, $b=3$, $c=-5$.

11. Simplify $(a+b-c)-(b+c-a)-(c+a-b)-(a+b+c)$.

12. Multiply $1-4x-10x^2$ by $1-6x+3x^2$.

13. Find the product of $x+1$, $x+2$ and $x-3$.

14. Divide the product of $x+2$, $2x-3$, $3x-2$ by $3x^2+4x-4$ and check when $x=1$.

15.* Multiply $a^2+b^2+c^2-ab-bc-ca$ by $a+b+c$.

16. Find the product of $a-2$, $a+2$, a^2+4, a^4+16.

17. Divide x^4+64 by x^2+4x+8.

18. Divide $x^4+x^3-24x^2-35x+57$ by x^2+2x-3, using detached coefficients and verify by multiplication using the same method.

19. Find the coefficient of x^3 in the product of $3x^3-2x^2+7x-2$ and $2x^3+5x^2+11x+4$.

20. The expression $44x^4-83x^3-74x^2+89x+56$ is the product of two expressions of which $4x^2-5x-7$ is one. Find the other.

21 Divide $x^4+4x^3+6x^2+4x+1$ by x^2+2x+1 and check by substituting $x=10$.

22. Subtract ax^2+bx+c from ex^2+dx+f, collecting the coefficients of powers of x in the result.

23. Find the remainder on dividing x^6+6 by x^2-1.

24. Show that

$$(a-b)(x-a-b)+(b-c)(x-b-c)+(c-a)(x-c-a)=0.$$

25. Show that $(a-b)(b-c)(c-a)=a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)$.

26. If $a=x^2+2xy+2y^2$, $b=x^2-2xy+2y^2$, $c=x^4-y^4$, find the value of $ab-c$.

27. Subtract $2x-3(y+2z)$ from $3y-(8z-3x)$.

28. If $s=a+b+c$, find in terms of a, b, c the value of

$$a(s-a)+b(s-b)+c(s-c).$$

29. Arrange in descending powers of x,

$$c(ax-b)-x(a-b)+bx(x^2-cx).$$

30. When $a=5$, find the value of

$$2a-\{3a-(4b+2a)\}+5a-(4b-a).$$

31. What quantity when divided by x^2-2x+3 gives x^2+2x-3 as quotient and 9 as remainder ?

32. If $a=x^2-3x+2$, $b=3x^2-10x+8$, $c=4x^2-9x+2$, find the value of $(a+2b-c)\div(x-2)$.

33. Arrange in descending order of magnitude and find the average of: 30, -15, 27, 0, 3, -10, -2, 6, -8.

34. What number must be added to $5x^3-13x^2+2x-1$ so that the sum may be divisible by $x-2$?

35. Find the coefficient of x^3 when $1+x+x^2$ is divided by $1-x-x^2$.

36. Divide $3p^4-7p(1-p^2)-(2+p^2)$ by $(3p+1)(p+1)$.

37. If x^2-9x+c is divisible by $x+4$, find c.

38. Find the sum of the coefficients in the square of $2x^2-x-3$.

39. Find the product of $x+a$, $x+b$, $x+c$. Collect the coefficients of the powers of x in the product. From the result, write down the product of $x+1$, $x+2$, $x+3$ and of $x-1$, $x-3$, $x+4$.

40. Divide x^6-2x^3+1 by x^2-2x+1.

41. When $a=3$, $b=2\frac{1}{3}$, $c=2$, find the value of

$$\frac{a^2b}{7}+\sqrt{7ab(2c^2-ab)}-(2a-3b)^2.$$

42. Prove that $(1+x)^2(1+y^2)-(1+x^2)(1+y)^2=2(x-y)(1-xy)$.

43. If $p=x-\frac{1}{x}$ and $q=x^2-\frac{1}{x^2}$, show that $p^2(p^2+4)=q^2$.

44. Divide $a^3+b^3+c^3-3abc$ by $a+b+c$. What are the factors of $a^3+b^3+c^3-3abc$? Compare with Ex. 15.

45. Multiply x^2+bx+c by x^2+px+q, arranging the product in descending powers of x.

46. Divide $9a^2-4b^2-c^2+4bc$ by $3a-2b+c$.

47. Multiply $x^2-x(a-1)-1$ by x^2+ax+1.

48. Divide $a^4-16b^4c^4$ by $a-2bc$.

49. Arrange the product of $x-a$, $x-b$, $x-c$ in descending powers of x.

50. Divide $a^2b+b^2c+c^2a-ab^2-bc^2-ca^2$ by $a-b$, and divide the quotient by $a-c$.

51. What expression will give a quotient of x^2+1 and a remainder of $2x^2-7x+6$ when divided by $3x^3-10x^2-6$?

52. Divide $x^3-y^3+6y^2-12y+8$ by $x-y+2$.

CHAPTER XI

FACTORING (*continued*)

89. In Chapter VIII. we have already dealt with the subject of factoring in simple cases. This Chapter will furnish a review of the methods already used, and an extension of those methods to more difficult examples.

90. Type I. Factors common to every Term. When every term of an expression contains the same factor, that factor can be found by inspection (art. 58).

Thus, $2xy$ is a factor of $4x^2y-6xy^2+2axy$,

$$\therefore\quad 4x^2y-6xy^2+2axy=2xy(2x-3y+a).$$

Also, $x+y$ is a factor of $a(x+y)+b(x+y)$. When this expression is divided by $x+y$, the quotient is $a+b$,

$$\therefore\quad a(x+y)+b(x+y)=(x+y)(a+b).$$

91. Type II. Factors by Grouping. When every term has not a common factor, if the number of terms be changed by grouping, we may sometimes obtain a common factor.

Ex. 1.—Factor $mx+nx+my+ny$.

(1)	(2)
$mx+nx+my+ny,$	$mx+nx+my+ny,$
$=x(m+n)+y(m+n),$	$=m(x+y)+n(x+y),$
$=(m+n)(x+y).$	$=(x+y)(m+n).$

Here we changed from four terms to two, and we found a common factor in the two terms. The other factor of the expression was then found by division.

The two solutions show that different methods of grouping may be employed. If the first method tried is not successful, try others.

Usually those terms are grouped which contain a simple common factor. In the example we should not expect to be successful by grouping *mx* with *ny*, as these terms have not a common factor.

Ex. 2.—Factor x^3+x^2+2x+2.

Use two different methods of grouping and obtain the factors $x+1$ and x^2+2.

Ex. 3.—Factor $(a-b)^2-ax+bx$.

$$\begin{aligned}(a-b)^2-ax+bx&=(a-b)^2-(ax-bx),\\&=(a-b)^2-x(a-b),\\&=(a-b)(a-b-x).\end{aligned}$$

NOTE.—When quantities are enclosed in brackets, the pupil must not forget to verify by mentally removing the brackets.

EXERCISE 67 (1-9, Oral)

Factor:

1. $3x-27$.
2. $2a-6$.
3. a^2-3a.
4. b^2-5b.
5. $3a^2-15ab$.
6. $6x^2y-12xy^2$.
7. $a(x+y)+b(x+y)$.
8. $p(m-n)+(m-n)$.
9. $x(a-b)-2y(a-b)$.

Factor, using two different methods of grouping and verify by multiplication:

10. $ax+bx+ay+by$.
11. $am-bm+an-bn$.
12. $x^2-ax+bx-ab$.
13. $bx-ax+ab-x^2$
14. $2ac+3ad-2bc-3bd$.
15. x^3+x^2+x+1.
16. a^3-a^2-3a+3.
17. $x-y-xy+1$.
18. $x^3+4x^2-3x-12$.
19. $a^3-7a^2-4a+28$.

20. Factor $x^5+x^4-x^3-x^2+x+1$ by making three groups each containing two terms, also by making two groups each containing three terms.

21.* Find three factors of $3x^3-6x^2+3x-6$ and of $axy-ay-ax+a$.

22. Find a common factor of

$$am+bm+an+bn \text{ and } ax+ay+bx+by,$$

and of x^3-x^2+x-1 and x^3-x^2+2x-2.

23. Factor $10x^2-5xy-6xz+3yz$ and $a^3b+a^2bc-3a^2b^2-3ab^2c$.

24. Find a common factor of $2x^3-6x^2-3x+9$ and x^3-3x^2+2x-6, and show that it is a factor of their difference.

25. Factor $48ax-56ay-35by+30bx$.

26. Factor $(x+y)^2+4x+4y$ and $2(a-b)^2-a+b$.

92. Type III. Complete Squares. We have already seen in art. 64 how the square of a binomial may be written down. We have also seen in art. 65 how the square root of a trinomial may be found, when the trinomial is a perfect square.

93. Square of a Trinomial. A trinomial may be squared by expressing it in the form of a binomial or by multiplication.

	a	b	c
a	a^2	ab	ac
b	ab	b^2	bc
c	ac	bc	c^2

Thus,

$$(a+b+c)^2=\{a+(b+c)\}^2,$$
$$=a^2+2a(b+c)+(b+c)^2,$$
$$=a^2+2ab+2ac+b^2+2bc+c^2,$$
$$\therefore\ \boldsymbol{(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc.}$$

Multiply $a+b+c$ by $a+b+c$ in the ordinary way and compare the results. Examine the diagram and see that the same result is obtained.

Similarly, $(a+b-c)^2=\{a+(b-c)\}^2$,
and $(a-b+c)^2=\{a-(b-c)\}^2$.

Complete these two in a manner similar to the one worked in full.

If we examine these products we see that they consist of two kinds of terms, squares (a^2, b^2, c^2) and double products ($2ab$, $2ac$, $2bc$).

We might express the result thus:

The square of any expression is equal to the sum of the squares of each of its terms, together with twice *the sum of the products of each pair of terms.*

In writing down the square, care must be taken to attach the proper sign to each double product.

Ex. 1. $(2x-3y+4z)^2$

$$=4x^2+9y^2+16z^2-12xy+16xz-24yz.$$

Ex. 2. $(a-2b+c-d)^2$

$$=a^2+4b^2+c^2+d^2-4ab+2ac-2ad-4bc+4bd-2cd.$$

Ex. 3. Factor $x^2+4y^2+z^2-4xy-2xz+4yz$.

This is evidently the square of an expression of the form $x \pm 2y \pm z$. Which of these when squared will give the proper arrangement of signs? Verify by writing down the square.

EXERCISE 68 (1-33, Oral)

What are the squares of:

1. $m+n$.
2. $m-n$.
3. $3x+2$.
4. $3a-5$.
5. $x-2y$.
6. $4x-y$.
7. $2a-3b$.
8. $3m-5n$.
9. $x+\frac{1}{2}$.
10. $2a-\frac{1}{2}$.
11. $x+y-z$.
12. $x-y+z$.
13. $a-b-c$.
14. $a+b+c+d$.
15. $p-q-r+s$.
16. $x-y+z-1$.

Express as squares:

17. $x^2+4xy+4y^2$.
18. $x^2-6xy+9y^2$.
19. $4x^2+4x+1$.
20. $4a^2-20ab+25b^2$.
21. $9a^2-12ab+4b^2$.
22. $m^2n^2-8mn+16$.
23. $4a^2+2a+\frac{1}{4}$.
24. $1-10a+25a^2$.
25. $a^4+2a^2b^2+b^4$.

What are the square roots of:

26. $x^2y^2+10xyz+25z^2$.
27. $16x^2-24xy+9y^2$.
28. $4-20a^2+25a^4$.
29. $(a+b)^2-2c(a+b)+c^2$.
30. $m^2+n^2+p^2+2mn+2mp+2np$.
31. $a^2+b^2+c^2-2ab+2ac-2bc$.
32. $x^2+4y^2+z^2+4xy+2xz+4yz$.
33. $4a^2+b^2+9c^2-4ab-12ac+6bc$.

Simplify:

34.* $(3x-y)^2+(x-3y)^2+(2x+3y)^2$.

35. $(a-b)^2+(b-c)^2+(c-a)^2+(a+b+c)^2$.

36. $(x^2+x+1)^2+(x^2-x+1)^2$.

37. $(a-b+c)^2+(b-c+a)^2+(c-a+b)^2$.

38. $(3x-2y+z)^2-(x-2y+3z)^2$.

Complete the squares by supplying the missing terms:

39. x^2- . . . . . $+25$.

40. $4x^2+$ . . . . . $+25y^2$.

41. a^2+4ab . . . . . .

42. . . . . . $-12mn+9n^2$.

43. a^2+9b^2+ . . . . . $-6ab-2ac$ . . . . . .

44. $9x^2+$ . . . . . $+$ . . . . . $-6xy-12xz$ . . . . . .

45. Find three factors of $3x^2+6x+3$ and of $a^3+4a^2b+4ab^2$.

46. Factor

$$(a+b)^2+4c(a+b)+4c^2 \text{ and } (a+b)^2-2(a+b)(c+d)+(c+d)^2.$$

47. Show that the square of the sum of any two consecutive integers is less than twice the sum of their squares by unity.

48. Divide the sum of the squares of $a-2b+c$, $b-2c+a$, $c-2a+b$ by the sum of the squares of $a-b$, $b-c$, $c-a$.

49. If $x+\frac{1}{x}=4$, find the value of $x^2+\frac{1}{x^2}$.

50. Factor $(ax+by)^2+(bx-ay)^2+c^2(x^2+y^2)$.

51. Express $a^2x^2+b^2y^2+a^2y^2+b^2x^2$ as the sum of two squares.

52. Find the value of $x^2+y^2+z^2+2xy+2xz+2yz$, when

$$x=a+2b-3c,\ y=b+2c-3a,\ z=c+2a-3b.$$

94. Type IV. The Difference of Squares. The product of the sum and difference of the same two quantities is equal to the difference of their squares (art. 66).

Conversely, the difference of the squares of two quantities is equal to the product of their sum and difference (art. 67).

Or, in symbols, $(a+b)(a-b)=a^2-b^2$,

and $a^2-b^2=(a+b)(a-b)$.

95. The formula for the product of the sum and difference may sometimes be used to find the product of expressions of more than two terms.

Ex. 1.—Multiply $2a-b+c$ by $2a-b-c$.

Here $2a-b+c$ is the sum of $2a-b$ and c,

and $2a-b-c$ is the difference of $2a-b$ and c.

They might be written $(2a-b)+c$ and $(2a-b)-c$. The produc is therefore the difference of the squares of $2a-b$ and c.

$$\therefore\quad (2a-b+c)(2a-b-c)=(2a-b)^2-c^2,$$
$$=4a^2-4ab+b^2-c^2.$$

Ex. 2.—Find the product of $2x+y-z$ and $2x-y+z$.

Here the first expression $=2x+(y-z)$,
and the second $=2x-(y-z)$,
$\therefore$ the product $=(2x)^2-(y-z)^2$,
$=4x^2-(y^2-2yz+z^2)$,
$=4x^2-y^2+2yz-z^2$.

Verify by ordinary multiplication.

Ex. 3.—Multiply $a-b+c-d$ by $a+b-c-d$.

Note that the terms with the same signs in the two expressions are a and $-d$. These should be grouped to form the first term in each factor.

$$a-b+c-d=(a-d)-(b-c),$$
$$a+b-c-d=(a-d)+(b-c),$$
$$\therefore\ \text{the product}=(a-d)^2-(b-c)^2.$$

Simplify this result and verify by multiplying in the ordinary way.

Ex. 4.—Factor $p^2-4pq+4q^2-x^2$.

Here the first three terms form a square and the expression may be written:

$$(p^2-4pq+4q^2)-x^2=(p-2q)^2-x^2,$$
$$=(p-2q+x)(p-2q-x).$$

What two quantities were here added and subtracted to obtain the factors?

Ex. 5.—Factor $a^2-b^2+2bc-c^2$.

Here the last three terms should be grouped to form the second square.

$$\therefore\quad a^2-b^2+2bc-c^2=a^2-(b^2-2bc+c^2),$$
$$=a^2-(b-c)^2,$$
$$=\{a+(b-c)\}\{a-(b-c)\},$$
$$=(a+b-c)(a-b+c).$$

Verify by multiplication.

Ex. 6.—Factor $x^2+y^2-a^2-b^2+2xy+2ab$.

Evidently three of these terms form one square and the remaining three the other square.

The expression $=(x^2+2xy+y^2)-(a^2-2ab+b^2)$,

$$=(x+y)^2-(a-b)^2,$$
$$=(x+y+a-b)(x+y-a+b).$$

EXERCISE 69 (1-10, 17-32, Oral)

Use the formula to obtain the following products:

1. $(2a+3)(2a-3)$. 2. $(4x-1)(4x+1)$.
3. $(xy+5)(xy-5)$. 4. $(ab-c)(ab+c)$.
5. $(2m^2+3n)(2m^2-3n)$. 6. $(abc+xy)(abc-xy)$.
7. $(x+\frac{1}{2})(x-\frac{1}{2})$. 8. $(x^2-y^2)(x^2+y^2)$.
9. $(x+y+z)(x+y-z)$. 10. $(a-b-c)(a-b+c)$.
11.* $(a+b-c)(a-b+c)$. 12. $(2x+3y-5)(2x+3y+5)$.
13. $(p-2q+3r)(p+2q-3r)$. 14. $(1-x+x^2)(1+x+x^2)$.
15. $(a+b-c+d)(a-b-c-d)$.
16. $(a-2b+c-2d)(a-2b-c+2d)$.

Factor and verify:

17. x^2-9. 18. $4x^2-25$. 19. a^2-4b^2.
20. $a^2b^2-x^2$. 21. $16x^2-9y^2$. 22. 93^2-4^2.
23. $1-a^2b^2$. 24. $25-x^4$. 25. $(a-b)^2-c^2$.
26. $(x+y)^2-25$. 27. $c^2-(a+b)^2$. 28. $x^2-(y-z)^2$.
29. $(a+b)^2-(c-d)^2$. 30. $(a+2b)^2-4c^2$. 31. $x^2+2xy+y^2-a^2$.
32. $a^2-2ab+b^2-c^2$. 33. $a^2-b^2-c^2-2bc$. 34. $a^2-b^2-4c^2+4bc$
35. $(4x+3)^2-16x^2$. 36. $1-x^2+2xy-y^2$. 37. $a^2-x^2+2ay+y^2$.
38. $a^2+b^2+2ab-c^2-d^2-2cd$. 39. $a^2-b^2+c^2-d^2-2ac-2bd$.
40. $a^2-2a+1-b^2+2bc-c^2$. 41. $x^4-x^2-4-2x^2y^2-4x+y^4$.
42. $4x^2-4x-y^2+4ay-4a^2+1$. 43. $1+a^2-b^2-4c^2+4bc-2a$.
44. Find three factors of $2x^2-8$, a^3-a, a^4-x^4.

45. Find three factors of $5a^2-10ab+5b^2-20c^2$ and of

$$(x-3b)^3-4b^2x+12b^3.$$

46. Find four factors of $a^2b^2-a^2c^2-b^2d^2+c^2d^2$ and of

$$(a^2+b^2-c^2)^2-4a^2b^2.$$

47. Factor $a^2x^2-b^2y^2+2acx+c^2$ and $m^2-9m^2n^2+n^2-2mn$.

48. Find the simplest factors of $3x^3-2x^2-3x+2$ and of

$$x^4-x^3-9x^2+9x.$$

49. Simplify $(a-b+c)(a-b-c)+(a+b-c)(a-b+c)$.

50. Arrange $x^2(x^2-a^2)-y^2(y^2-a^2)+2xy(x^2-y^2)$ so as to show that x^2-y^2 is a factor of it, and thus find the simplest factors.

51. Use factoring to simplify:

(1) $(a^2-3a+1)^2-(a^2-3a)^2$.
(2) $(x-2y+3z)^2-(3z-x+2y)^2$.
(3) $(a^2-3a-4)^2-(a^2-4)^2$.
(4) $(5x^2-2xy+y^2)^2-(5x^2+2xy-y^2)^2$.

52. Multiply $a+b+c$ by $a+b-c$ and $a-b+c$ by $a-b-c$ and use the results to obtain the product of

$$(a+b+c)(a+b-c)(a-b+c)(a-b-c).$$

53. Show that $x(y^2-z^2)+y(z^2-x^2)+z(x^2-y^2)$ is equal to

$$(x-y)(y^2-z^2)-(x^2-y^2)(y-z)$$

and then find the factors of this expression.

54. Arrange $a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)$ in the form

$$a(b^2-c^2)-bc(b-c)-a^2(b-c)$$

and thus obtain the factor $b-c$. Find the other two factors.

96. Type V. Incomplete Squares. We have already factored many expressions which were seen to be the difference of two squares.

Sometimes the two squares of which an expression is the difference are not so easily seen.

Ex. 1.—Factor $x^4+x^2y^2+y^4$.

This expression would be the square of x^2+y^2 if the middle term were $2x^2y^2$. We will therefore add x^2y^2 to complete the square and also subtract x^2y^2 to preserve the value of the expression.

Then
$$\begin{aligned} x^4+x^2y^2+y^4 &= x^4+2x^2y^2+y^4-x^2y^2, \\ &= (x^2+y^2)^2-(xy)^2, \\ &= (x^2+y^2+xy)(x^2+y^2-xy). \end{aligned}$$
$$\therefore \quad x^4+x^2y^2+y^4=(x^2+xy+y^2)(x^2-xy+y^2).$$

In order that this method may be successful, it will be seen that the quantity we add to complete the square must itself be a square.

Thus, to change a^2+ab+b^2 into $a^2+2ab+b^2-ab$ is of no value as ab is not an algebraic square.

Ex. 2.—Factor a^4+4b^4.

This can be made the square of a^2+2b^2 by adding $4a^2b^2$.

$$\therefore \quad a^4+4b^4=a^4+4a^2b^2+4b^4-4a^2b^2.$$

Complete the factoring and verify by multiplication.

Ex. 3.—Factor $4m^4-16m^2n^2+9n^4$.

What must be added to make it the square of $2m^2-3n^2$? Complete the factoring.

Try to factor it by making it the square of $2m^2+3n^2$.

Ex. 4.—Factor $a^4+b^4+c^4-2a^2b^2-2b^2c^2-2c^2a^2$.

How does this expression differ from the square of $a^2+b^2-c^2$?
Express it in the form $(a^2+b^2-c^2)^2-4a^2b^2$.
Write down the two factors and see if you can factor each of them again and thus obtain the result

$$(a+b+c)(a+b-c)(a-b+c)(a-b-c).$$

EXERCISE 70

Factor and verify:

1. a^4+a^2+1.
2. x^4+x^2+25.
3. x^4+7x^2+16.
4. $x^4+2x^2y^2+9y^4$.
5. $4a^4+1$.
6. $9x^4+8x^2y^2+16y^4$.
7. $4b^4-13b^2+1$.
8. $9a^4-15a^2+1$.
9. $9a^4-52a^2b^2+64b^4$

10. $25x^4-89x^2y^2+64y^4$. 11. $x^4+y^4-11x^2y^2$. 12. x^8-7x^4+1.

13.* Find three factors of $2x^4+8$ and x^5+x^3+x.

14. Find four factors of $9a^4-10a^2b^2+b^4$.

15. Find three factors of x^8+x^4+1.

16. Find four factors of $a^4+b^4+c^4-2a^2b^2-2b^2c^2-2c^2a^2$ by completing the square of $a^2-b^2+c^2$.

17. Factor $(a+1)^4+(a^2-1)^2+(a-1)^4$.

97. Type VI. Trinomials. We have already dealt with the factoring of expressions of the type x^2+px+q, where the coefficient of the first term is unity (art. 61).

We now wish to factor expressions of the type mx^2+px+q, where m is not necessarily unity.

98. First Method, by Cross Multiplication.

Ex. 1.—Factor $2x^2+7xy+3y^2$.

The product of the first terms of the factors is $2x^2$, and therefore the first terms must be $2x$ and x; similarly, the last terms must be $3y$ and y and the signs are evidently all positive.

$\therefore$ the factors must be

$$\begin{array}{r} 2x+3y \\ x+\ y \\ \hline \end{array} \qquad \text{or} \qquad \begin{array}{r} 2x+\ y \\ x+3y \\ \hline \end{array}$$

It is seen, by cross multiplication, that the coefficient of xy in the first product is $3+2=5$, and in the second is $1+6=7$.

$\therefore$ the correct factors are $(2x+y)(x+3y)$.

Ex. 2.—Factor $3x^2-7x-6$.

Here the numerical coefficients of the first terms of the factors must be 3 and 1, and of the last terms may be 6 and 1 or 3 and 2.

Since the third term is negative, the signs of the second terms of the factors must be different.

The possible sets of factors, omitting the signs, are :

$$\begin{array}{rr} 3x & 3 \\ x & 2 \\ \hline \end{array}, \qquad \begin{array}{rr} 3x & 2 \\ x & 3 \\ \hline \end{array}, \qquad \begin{array}{rr} 3x & 6 \\ x & 1 \\ \hline \end{array}, \qquad \begin{array}{rr} 3x & 1 \\ x & 6 \\ \hline \end{array}.$$

Since the signs are different for the last terms, when we cross multiply to find the coefficient of x in the product, the partial products must be subtracted.

It is easily seen that the second arrangement is the only one from which $7x$ can be obtained.

Since the middle term is negative, the larger of the cross products must be negative.

$\therefore$ the factors are $(3x+2)(x-3)$.

This method is liable to be found tedious when the coefficients have a number of pairs of factors, but in ordinary cases the pupil will find little difficulty after he has had some practice in the work.

99. Second Method, by Decomposition. In the process of multiplying two binomials like $2x+3$ and $3x+5$, we have

$$\begin{aligned}(2x+3)(3x+5)&=3x(2x+3)+5(2x+3),\\&=6x^2+9x+10x+15,\\&=6x^2+19x+15.\end{aligned}$$

If we wish to factor a trinomial like $6x^2+19x+15$, we may do so by reversing the process.

Thus,
$$\begin{aligned}6x^2+19x+15&=6x^2+9x+10x+15,\\&=3x(2x+3)+5(2x+3),\\&=(2x+3)(3x+5).\end{aligned}$$

The only difficulty in this method is in finding the two terms into which the middle term, $19x$, should be decomposed. This difficulty may be overcome in the following way:

$$(ax+b)(cx+d)=acx^2+x(ad+bc)+bd.$$

Note that the product of the two terms in the coefficient of x, ad and bc, is the same as the product of the coefficient of x^2, ac and the absolute term, bd.

In the trinomial $6x^2+19x+15$ above, the product of 6 and 15 is 90 and the two factors of 90 whose sum is 19 are 9 and 10, which shows that the middle term, $19x$, should be decomposed into $9x+10x$.

Ex. 1.—Factor $6x^2+13x+6$.

The product of the coefficient of x^2 and the absolute term is 36. The two factors of 36 whose sum is 13 are 4 and 9.

$$\begin{aligned}\therefore\quad 6x^2+13x+6 &= 6x^2+4x+9x+6,\\ &= 2x(3x+2)+3(3x+2),\\ &= (3x+2)(2x+3).\end{aligned}$$

Ex. 2.—Factor $12x^2-17x-5$.

Here we require two factors of -60 whose sum is -17, and they are evidently -20 and 3.

$$\begin{aligned}\therefore\quad 12x^2-17x-5 &= 12x^2-20x+3x-5,\\ &= 4x(3x-5)+(3x-5),\\ &= (3x-5)(4x+1).\end{aligned}$$

EXERCISE 71 (1-18, Oral)

Factor and verify:

1. x^2+4x+3.
2. $a^2+11a+30$.
3. $y^2+8y+15$.
4. $a^2-11a+18$.
5. $x^2-14x+48$.
6. $1+5x+6x^2$.
7. $x^2-15x+14$.
8. $a^2b^2-5ab+6$.
9. $a^2-15a+56$.
10. $1-21x+38x^2$.
11. $x^2-6xy+8y^2$.
12. $a^2-13ab+36b^2$.
13. x^2-4x-5.
14. $a^2-9a-22$.
15. $x^2-28x-29$.
16. $y^2-4y-21$.
17. $1-2a-15a^2$.
18. $a^2-ay-2y^2$.
19. $2x^2+5x+3$.
20. $4x^2+8xy+3y^2$.
21. $9a^2-18ab+8b^2$.
22. $8x^2+x-9$.
23. $3x^2-x-2$.
24. $6a^2-a-2$.
25. $4x^2+x-5$.
26. $15b^2-19b-8$.
27. $10x^2-23x-5$.
28. $10b^2-89b-9$.
29. $9x^2-31xy+12y^2$.
30. $10a^2-29ab+10b^2$.

Find the simplest factors of:

31.* $3x^2-3x-216$.
32. $2a^2+8a+6$.
33. $x-5x^2+6x^3$.
34. x^4-5x^2+4.
35. a^5-10a^3+9a.
36. $9a^4-10a^2+1$.
37. $(x^2+4x)^2-2(x^2+4x)-15$.
38. $(x^2-9x)^2+4(x^2-9x)-140$.
39. Without multiplying show that

$$(x^2-x-2)(x^2+2x-15)=(x^2+6x+5)(x^2-5x+6).$$

40. An expression is divisible by $x-2$, the quotient being x^2-x-6. Show that it is divisible by $x+2$ and find the quotient.

41. Show that the product of $6x^2-13x+6$ and $2x^2-7x+5$ is divisible by $3x^2-5x+2$ and find the quotient.

42. If $3x^2+ax-14$ is the product of two binomials with integral coefficients, find all the different values that a may have.

43. By factoring, find the quotient when the product of

$$6a^2+7ab-20b^2 \text{ and } 22a^2-73ab-14b^2$$

is divided by $4a^2-4ab-35b^2$.

44. Factor $x^2+5xy+4y^2+x+y$.

45. Factor $3a^2-ab-2b^2+6a+4b$.

46. Divide the product of x^2+3x+2 and x^4-1 by the product of x^2+2x+1 and x^2+x-2.

100. Type VII. Sum and Difference of Cubes.

Divide x^3+y^3 by $x+y$,
and x^3-y^3 by $x-y$.

$\therefore$ $x^3+y^3=(x+y)(x^2-xy+y^2)$,
and $x^3-y^3=(x-y)(x^2+xy+y^2)$.

Examine carefully the signs in these factors.

$$\begin{array}{l} x+y)x^3 \qquad\qquad +y^3(x^2-xy+y^2 \\ \quad\ \underline{x^3+x^2y} \\ \qquad -x^2y \\ \qquad \underline{-x^2y-xy^2} \\ \qquad\qquad +xy^2+y^3 \\ \qquad\qquad\ \ \underline{xy^2+y^3} \end{array}$$

It is thus seen, that *the sum of the cubes of two quantities is divisible by their sum, and the difference of the cubes is divisible by their difference.*

The quotient in each case consists of the square, product and square of the terms of the divisor, with the proper algebraic signs.

Ex. 1.—Factor $8a^3+27b^3$.

Here $8a^3=(2a)^3$ and $27b^3=(3b)^3$,

$\therefore$ the expression may be written $(2a)^3+(3b)^3$,

$\therefore$ the first factor is $2a+3b$ and the second is

$$(2a)^2-(2a)(3b)+(3b)^2 \text{ or } 4a^2-6ab+9b^2.$$

$$\therefore\quad 8a^3+27b^3=(2a+3b)(4a^2-6ab+9b^2).$$

Ex. 2.—Factor $a^3x^3-64y^6$.

$$\begin{aligned} a^3x^3-64y^6 &= (ax)^3-(4y^2)^3, \\ &= (ax-4y^2)(a^2x^2+4axy^2+16y^4). \end{aligned}$$

Ex. 3.—Factor x^6-y^6.

This may be expressed as the difference of two squares or of two cubes.

$$\begin{aligned}\therefore\quad x^6-y^6&=(x^3)^2-(y^3)^2, \qquad \text{or } (x^2)^3-(y^2)^3,\\ &=(x^3+y^3)(x^3-y^3), \text{ or } (x^2-y^2)(x^4+x^2y^2+y^4).\end{aligned}$$

Complete the factoring by each method and decide which you will use, if you have the choice, as here.

EXERCISE 72 (1-12, Oral)

State one factor of:

1. a^3+b^3. 2. x^3+8. 3. x^3-27. 4. $1000-a^3$.
5. x^3-64y^3. 6. $27-b^3$. 7. $8a^3+125$. 8. $125a^3-8b^3$.
9. $1-27x^3$. 10. $343x^3-8$. 11. $(a+b)^3+c^3$.
12. $(a-b)^3-c^3$.

Factor and verify 13-21:

13. a^3+27. 14. x^3-8y^3. 15. $8a^3+1$.
16. $27x^3-64y^3$. 17. $8-27a^3$. 18. $1000x^3-y^3$.
19. a^3+b^6. 20. x^6-b^3. 21. a^3-y^6.
22.* $2a^3-16$. 23. $81+3y^3$. 24. a^4+a.
25. a^3b+b^4. 26. a^6+b^6. 27. $(x+y)^3+a^3$.
28. $(x-2)^3+8$. 29. $(a-b)^3+a^3$. 30. $(a-b)^3+(a+b)^3$.

31. What is one factor of $(2x-y)^3-(x-2y)^3$?

32. Show that $(2a-3b)^3+(3a-2b)^3$ is divisible by $a-b$.

33. Factor $(a^2-2bc)^3+8b^3c^3$ and $27x^3y^3z-y^6z^4$.

34. Find six factors of $a^{12}-b^{12}$.

35. Find two binomial factors of $(2x^2-3x+3)^3-(x^2-2x+5)^3$.

36. If $x+\dfrac{1}{x}=2$, find the value of $x^3+\dfrac{1}{x^3}$.

37. By factoring show that $(a+b)^4-3ab(a+b)^2=(a+b)(a^3+b^3)$.

101. Type VIII. The Factor Theorem. What are the values of 0×5, $a\times0$, $0\times(-4)$, $\frac{1}{2}\times0$, -1000×0?

If one of the factors of a product be zero, the product must also be zero.

If the product of two numbers be zero, what can we infer?

If $ab=0$, it follows that either $a=0$ or $b=0$.

If $(x-3)(x-4)=0$, then either $x-3=0$ or $x-4=0$.

Since $(x-2)(x^2-7x+12)=x^3-9x^2+26x-24$,

$\therefore$ $x^3-9x^2+26x-24$ must be equal to zero when $x=2$, for then one of its factors, $x-2$, is zero.

If we substitute 2 for x, we see that this is true.

$$\begin{aligned} x^3-9x^2+26x-24 &= 2^3-9\,.\,2^2+26\,.\,2-24, \\ &= 8 - 36 + 52 - 24 = 0. \end{aligned}$$

Conversely, when any expression becomes zero when $x=a$, then $x-a$ is a factor of it.

Substitute $x=3$ in $x^3-6x^2+11x-6$ and it becomes

$$3^3-6\,.\,3^2+11\,.\,3-6=27-54+33-6=0,$$

$$\therefore \quad x-3 \text{ is a factor of } x^3-6x^2+11x-6.$$

Divide it by $x-3$ and the other factor is x^2-3x+2.

$$\begin{aligned} \therefore \quad x^3-6x^2+11x-6 &= (x-3)(x^2-3x+2), \\ &= (x-3)(x-2)(x-1). \end{aligned}$$

If $x+1$ is a factor of an expression, the expression must be equal to zero when $x=-1$, for then $x+1=0$.

Thus, $x+1$ is a factor of $x^3-x^2-10x-8$, since

$$(-1)^3-(-1)^2-10(-1)-8=-1-1+10-8=0.$$

Divide by $x+1$ and complete the factoring.

Any expression is divisible by x−a *if it vanishes (becomes zero) when* a *is substituted for* x.

This is called the **factor theorem**.

Show that $x-a$ is a factor of $x^3-7ax^2+10a^2x-4a^3$.

Show that $x+a$ is a factor of $5x^3+6x^2a+11xa^2+10a^3$.

Ex.—Factor $x^3-9x+10$.

If it has a binomial factor it must be of the form

$$x\pm1, \quad x\pm2, \quad x\pm5 \text{ or } x\pm10.$$

Testing for these factors we find that $x-2$ is a factor,

$$\therefore \quad x^3-9x+10=(x-2)(x^2+2x-5).$$

The factoring is complete as x^2+2x-5 has no simple factors.

102. Special Case. It is easy to see when $x-1$ is a factor of any expression, for when 1 is substituted for x, the value of the expression becomes equal to the sum of its coefficients.

Thus, if $x=1$

$$\begin{aligned} &x^3-2x^2-19x+20,\\ &=1-2-19+20=0, \end{aligned}$$

$\therefore x-1$ is a factor. Complete the factoring.

Similarly, $x-a$ is a factor of $3x^3-16x^2a-7xa^2+20a^3$, since $3-16-7+20=0$, and $a-b$ is a factor of $a^3-6a^2b+3ab^2+2b^3$, since $1-6+3+2=0$.

EXERCISE 78

Each of these expressions is divisible by $x-1$, $x-2$ or $x-3$. Find all the factors of each and verify.

1. $x^3-10x^2+29x-20$.
2. $x^3-3x^2-12x+14$.
3. $x^3+5x^2-2x-24$.
4. x^3-4x^2+x+6.
5. $2x^3-7x^2+7x-2$.
6. $4x^3-9x^2-10x+3$.

Factor:

7.* $2x^3-11x^2+5x+4$.

8. x^3-2x^2-x+2.

9. x^3-7x+6.

10. $x^3-19x+30$.

11. $a^3+a^2-10a+8$.

12. $a^3-3ab^2-2b^3$.

13. Show that $x+2$ is a factor of x^3-x^2-x+10.

14. Show that $x+a$ is a factor of $x^3+7x^2a+9xa^2+3a^3$.

15. Show that $x+3$, $x+4$ and $x-7$ are the factors of $x^3-37x-84$.

16. If $x^3-10x+a$ is divisible by $x+2$, find a.

17. Show that $a-b$ is a factor of $a^3+4a^2b+ab^2-6b^3$, and find all the factors.

18. Noting that $x^2-2x-3=(x+1)(x-3)$, show that x^2-2x-3 is a factor of $x^4-4x^3+2x^2+4x-3$.

19. Show that $a-b$, $b-c$ and $c-a$ are factors of

$$a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2).$$

20. If $x-1$ and $x-2$ are factors of x^3-5x^2+ax+b, find a and b.

21. If $px^3-3x^2+qx-10$ and $qx^3+2x^2-17x+p$ are both divisible by $x-2$, find p and q.

103. Equations Solved by Factoring. We have seen that if

$$(x-3)(x-4)=0,$$

then $$x-3=0 \text{ or } x-4=0.$$

Thus the equation $(x-3)(x-4)=0$ is equivalent to the two simple equations $x-3=0$ and $x-4=0$.

But if $$x-3=0,\ x=3,$$
and if $$x-4=0,\ x=4,$$

$\therefore$ the roots of the equation $(x-3)(x-4)=0$ are 3, 4.

The truth of this may be seen by substitution.

If $x=3$, $(x-3)(x-4)=(3-3)(3-4)=0\times-1=0.$

If $x=4$, $(x-3)(x-4)=(4-3)(4-4)=1\times\ 0=0.$

Since $$(x-3)(x-4)=x^2-7x+12,$$
the given equation may be written

$$x^2-7x+12=0.$$

104. Quadratic Equation. Any equation which contains the square of the unknown and no higher power is called a **quadratic equation** or an equation of the second degree.

The preceding shows that if we wish to solve a quadratic equation we may do so by finding, by factoring, the simple equations of which it is composed.

The simple equations when solved will give the roots of the given quadratic equation.

Ex. 1.—Solve $$x^2-6x-7=0.$$

Factoring, $$(x-7)(x+1)=0,$$
$$\therefore\ x-7=0 \text{ or } x+1=0,$$
$$\therefore\ x=7 \text{ or } -1.$$

Verification : if $x=7$, $x^2-6x-7=49-42-7=0$,
if $x=-1$, $x^2-6x-7=\ 1+\ 6-7=0.$

Ex. 2.—Solve $$3x^2+7x=6.$$

Transpose the 6 so as to make the right-hand side zero, as in the previous problem

$$\therefore\ 3x^2+7x-6=0,$$
$$\therefore\ (3x-2)(x+3)=0,$$
$$\therefore\ 3x-2=0 \text{ or } x+3=0,$$
$$\therefore\ x=\tfrac{2}{3} \text{ or } -3.$$

Verify both of these roots.

Ex. 3.—Form the equation whose roots are 2 and −5.

The required equation is at once seen to be a combination of the two simple equations

$$x-2=0 \text{ and } x+5=0,$$

and therefore is

$$(x-2)(x+5)=0,$$

or

$$x^2+3x-10=0.$$

Ex. 4.—If 2 is a root of the equation

$$x^3+3x^2-16x+12=0,$$

find the other roots.

Since 2 is a root, then $x-2$ is a factor of $x^3+3x^2-16x+12$ and the other factor, found by division, is x^2+5x-6.

$$\therefore\ x^3+3x^2-16x+12=(x-2)(x-1)(x+6)=0,$$

$$\therefore\ x-2=0 \text{ or } x-1=0 \text{ or } x+6=0,$$

$$\therefore\ x=2 \text{ or } 1 \text{ or } -6.$$

$\therefore$ the other roots are 1 and −6.

EXERCISE 74 (1-16, Oral)

To what equations of the first degree is each of the following equivalent:

1. $(x-1)(x-2)=0.$
2. $(x-3)(x+5)=0.$
3. $x(x-5)=0.$
4. $(x-1)(x-2)(x-3)=0.$
5. $x^2-4=0.$
6. $x^2-4x+3=0.$
7. $x^2+5x+6=0.$
8. $x^2-x-20=0.$
9. $x^2+3ax+2a^2=0.$
10. $x^2-bx-12b^2=0.$

State the equations whose roots are:

11. 2 and 3.
12. 4 and −5.
13. −2 and −4.
14. a and b.
15. 2, 3 and 1.
16. 4, 5 and −6.

Solve and verify:

17. $x^2-8x+15=0.$
18. $x^2+8x+15=0.$
19. $x^2+2x-15=0.$
20. $x^2-2x-15=0.$
21. $3x^2-8x+4=0.$
22. $4x^2-2x-2=0.$
23. $2x^2+x=15.$
24. $x(3x-1)=10.$
25. $x^3-x=0.$
26. $x^2-ax+bx-ab=0.$

27. If 2 is a root of $x^3-19x+30=0$, find the other roots.

28. Solve $x^3-6x^2+11x-6=0$ and $4x^3-12x^2+11x-3=0$

(Note that the sum of the coefficients is zero.)

29. The sum of a number and its square is 42. Find the number.

30. The sum of the squares of two consecutive numbers is 61. Find them.

31. The sides of a right-angled triangle are x, $x+1$ and $x+2$. Find x.

105. Notes concerning Factoring. The subject of factoring is one of the important parts of algebra, as it enters into so many other processes. We have already had examples of its use in solving equations and in performing operations on fractions.

In the preceding exercises, in this Chapter, the expressions to be factored have been classified for the pupil. In the practical use of factoring, however, he must determine for himself the particular method to be used.

This is usually done by determining the **type** or **form** to which the expression belongs. The examples in the review exercise which follows will give the required practice.

The types which have been discussed in this Chapter are here collected for reference :

I. $ax+ay$. (Common factor in every term.)
II. $ax+ay+bx+by$. (Factored by grouping.)
III. $x^2\pm 2xy+y^2$. (Complete squares.)
IV. a^2-b^2. (Difference of two squares.)
V. $x^4+x^2y^2+y^4$. (Incomplete squares.)
VI. ax^2+bx+c. (Trinomials.)
VII. $x^3\pm y^3$. (Sum or difference of cubes.)
VIII. Factored by the factor theorem.

EXERCISE 75 (Review of Chapter XI)

1. State the squares of $a+b$, $a-b$, $x-3y$, $2x-1$, $3x-5$, $5a+2b$, $3x-4y$, $7a-3$, a^2-1, $a+\frac{1}{a}$.

2. State the squares of $a+b+c$, $x+y-z$, $a-b-c$.

3. Write down the products of $x(a-b)$, $a(a-b+c)$, $(x+1)(x+7)$, $(x-3)(x-5)$, $(2x-3)(2x+3)$, $(ax-6)(ax+7)$.

Use short methods to find, in the simplest form, the value of:

4.* $(x+a+b)(x+a-b)+(x-a-b)(x-a+b)$.

5. $(x^2+x+1)(x^2+x-1)-(x^2-x+1)(x^2-x-1)$.

6. $(a+b+c)^2+(a+b-c)^2+(a-b+c)^2+(a-b-c)^2$.

7. $(2a+3b-c)^2+(3a+b-2c)^2+(a-2b+3c)^2$.

8. 9999^2-9998^2.

9. 5743^2-4257^2.

10. $503\times497-502\times498$.

11. $(a+99)^2-(a+98)^2$.

Find the simplest factors and verify 12–29:

12. x^2-x-42.

13. x^3-3x^2-x+3.

14. x^3-4x.

15. a^2+a-56.

16. $x^2-a+ax-x$.

17. $27x^2-12y^2$.

18. $x^3+5x^2-4x-20$.

19. x^3-3x^2+2x.

20. $15a^2+32a+9$.

21. $343-x^3$.

22. x^4-4x^2.

23. $18x^2+48x+32$.

24. $(x-3)^2+(x-3)(x+4)$.

25. $15x^2-15y^2-16xy$.

26. $1+2ab-a^2-b^2$.

27. $x(x-2)+y(x-2)-x+2$.

28. $abc^2+a^2cd+abd^2+b^2cd$.

29. $25x^4y-40x^3y^2+16x^2y^3$.

30. $4(x-2)^2-x+2$.

31. $24a^4-3ab^3$.

32. $(a+2b-3c)^2-(3a+2b-c)^2$.

33. $12x^2-x-20$.

34. $108a^3-500$.

35. x^2+x-y^2+y.

36. x^4-7x^2-18.

37. $x^3-y^3-2x^2y+2xy^2$.

38. $x^2-xy-132y^2$.

39. $(a+c)(a-c)-b(2a-b)$.

40. $x^3+y^3+3xy(x+y)$.

41. $ax^2-x(3ab+2)+6b$.

42. $a^2-4b^2-3a-6b$.

43. $2x(2x+a)-y(y+a)$.

44. $a^2+2ab+b^2+ac+bc$.

45. $a^2-2ab+b^2-a+b$.

46. $x^3+x^2+x-y^3-y^2-y$. 47. $a^4b-a^2b^3-a^3b^2+ab^4$.

48. $4a^2-25b^2+2a+5b$. 49. $8(a+b)^3-(2a-b)^3$.

50. $x^4+y^4-23x^2y^2$. 51. $a^4-a^2-9-2a^2b^2+b^4+6a$.

52. x^3-11x^2+7x+3. 53. $3a^3-5a^2-6a+10$.

54. $x^2c^2-c^2+x^2-1$. 55. $a^7-a^3+8a^4-8$.

56. Show that $a-b+c$ is a factor of

$$(2a-3b+4c)^3+(2a-b)^3.$$

57. Factor $4a^4-37a^2b^2+9b^4$, (1) by cross multiplication, (2) by completing the square of $2a^2-3b^2$, (3) by completing the square of $2a^2+3b^2$.

58. Without multiplication show that

$$(x^2-4x+3)(x^2-12x+35)=(x^2-6x+5)(x^2-10x+21).$$

59. Make a diagram to show the square of $a+b+c+d$.

60. Factor $(a-b)(b^2-c^2)-(b-c)(a^2-b^2)$.

61. Find the factors of $6x^3-7x^2-16x+12$, being given that it vanishes when $x=2$.

62. Find four factors of $(x^2-5x)^2+10(x^2-5x)+24$ and of

$$(x^2-6)^2-4x(x^2-6)-5x^2.$$

63. Use the factor theorem to solve

$$x^3-31x+30=0 \text{ and } x^4-43x^2+42x=0.$$

64. If two numbers differ by 6, show that the difference of their squares is equal to six times their sum.

65. Find the quotient when the product of $x^2-(b-c)x-bc$ and $x^2-(c-a)x-ca$ is divided by $x^2+(a-b)x-ab$.

66. Multiply $a^2-b^2-c^2+2bc$ by $\dfrac{a+b+c}{a+b-c}$.

67. If $x^4+x^3+ax^2+bx-3$ is divisible by $x-1$ and $x+3$, find a and b and the remaining factor.

68. Factor $2x^2-ax+bx-ab-a^2$.

69. Express $a^2b^2+c^2d^2-a^2c^2-b^2d^2$ as the difference of two squares in two different ways.

70. Factor $a^4+b^4+c^4-2a^2b^2-2b^2c^2-2c^2a^2$ by completing the square of $a^2-b^2-c^2$.

71. Find four factors of $(a^2-b^2-c^2+d^2)^2-4(ad-bc)^2$.

CHAPTER XII

SIMULTANEOUS EQUATIONS (*continued*)

106. In Chapter VII. the solution of simple examples of equations in two unknowns has been considered.

The method there followed was to make the coefficients of one of the unknowns numerically equal by multiplication, and then that unknown was eliminated by addition or subtraction.

Other methods of eliminating one of the unknowns are useful in certain cases.

107. Elimination by Substitution.

Ex.—Solve

$$x-2y=2, \qquad (1)$$
$$5x+7y=78. \qquad (2)$$

From (1), $x=2+2y$. (3)

Substituting this value of x in (2),

$$5(2+2y)+7y=78,$$
$$\therefore\ 10+10y+7y=78,$$
$$\therefore\ 17y=68,$$
$$\therefore\ y=4.$$

Substituting $y=4$ in (3), $x=10$.

Here we eliminated x by finding the value of x in terms of y from (1) and substituting that value in (2). We thus obtained an equation which contained only the unknown y.

This is called the method of **elimination by substitution.**

We might take the value of x from (2) and substitute in (1).

Thus from (2), $5x=78-7y$, $\therefore\ x=\dfrac{78-7y}{5}$.

Substituting in (1), $\dfrac{78-7y}{5}-2y=2$.

Complete the solution and verify the roots.

The value of y might have been found from either equation and substituted in the other.

Thus from (1), $2y = x - 2$, $\therefore\ y = \frac{x-2}{2}$.

Substituting in (2), $5x + \frac{7(x-2)}{2} = 78$.

Complete the solution.

Solve also by finding y from (2) and substituting in (1).

If the four solutions be compared it will be seen that, in this problem, the first is the simplest.

In solving equations with two unknowns, the pupil should examine them carefully and choose the unknown which he thinks will be the simpler to deal with.

108. Elimination by Comparison.

Ex.—Solve

$$2x - 3y = 7, \qquad (1)$$

$$3x + 5y = 39. \qquad (2)$$

From (1), $x = \frac{7+3y}{2}$, and from (2), $x = \frac{39-5y}{3}$,

$$\therefore\ \frac{7+3y}{2} = \frac{39-5y}{3},$$

$$\therefore\ 3(7+3y) = 2(39-5y).$$

Complete the solution and verify the roots.

Here we effected the elimination of x by comparing the values of x from the two equations.

This is called the method of **elimination by comparison.**

We might have compared the values of y obtained from the two equations. Solve it that way.

109. Three Methods of Elimination. We have illustrated three methods of elimination, by addition or subtraction, by substitution and by comparison. When no particular method is specified, the pupil is advised to use the first method as no fractions appear in it.

EXERCISE 76 (1-6, Oral)

State the value of each unknown in terms of the other in :

1. $x+y=5.$ 2. $x-y=3.$ 3. $x+2y=11.$

4. $3x-y=6.$ 5. $2x+3y=12.$ 6. $5x-4y=19.$

Solve by substitution and verify :

7. $x+2y=18,\ 2x+5y=41.$ 8. $3x+y=7,\ 4x+3y=11.$ 9. $2x$ [illegible] $=19,\ 5x$ [illegible] $=46.$

10. $2x-3y=14,\ x-5y=0.$ 11. $3x-4y=10,\ 2x+6y=11.$ 12. $8x+$ [illegible] $y=$ [illegible], [illegible] $x-21$ [illegible] $12.$

Solve by comparison and verify :

13. $x+3y=10,\ x+5y=14.$ 14. $2x+y=26,\ 3x-y=14.$ 15. $3x+4y=16,\ 4x-3y=3$

Solve by any method and verify :

16. $\frac{1}{3}x-\frac{1}{4}y=2,\ \frac{2}{3}x+\frac{3}{4}y=9.$ 17. $3x=2y,\ \frac{1}{2}x=\frac{2}{3}y-2.$ 18. $y=\frac{1}{2}x$ [illegible], $\frac{2}{3}x-\frac{1}{4}y=3\frac{1}{2}.$

19. $\dfrac{x}{2}=\dfrac{y}{3},\quad y-\dfrac{5x}{12}=13.$ 20. $\dfrac{3x}{5}+\dfrac{y}{3}=6,\quad x+\dfrac{3y}{4}=11\frac{3}{4}.$ 21. $3y-7x=x,\quad \dfrac{3x-1}{y}=1.$

22. $3x+2-\dfrac{y+7}{11}=2y+\dfrac{x+11}{7}=10.$

23. $x-5y+3=2x-8y+3=7x-10y+16.$

24. $(x-1)(y-2)-(y-3)(x+1)=17,$
$(x-3)(y-5)-(x-5)(y-3)=-22.$

25. $\cdot 1x+\cdot 21y+\cdot 52=\cdot 01x+\cdot 01y+3=0.$

26. $x+5=3(y-3),\quad \dfrac{5x-4}{11}+y=\dfrac{2x-5}{9}+19.$

27. $\dfrac{4x-3y-5}{4}=7x-2y-\dfrac{23}{6}=\dfrac{2x-9y}{2}.$

28. If the sum of two numbers is $\frac{4}{3}$ of the greater number, the difference of the numbers is how many times the less ?

110. Equations with three Unknowns.

Ex.—Solve

$$2x+3y-4z=12, \qquad (1)$$
$$3x-\ y+2z=15, \qquad (2)$$
$$4x+\ y-3z=19. \qquad (3)$$

This system of equations differs from the preceding by containing **three unknown quantities.**

If we can obtain from these three equations, two equations containing the **same** two unknowns, the solution can be effected by preceding methods.

How can we obtain from (1) and (2) an equation containing x and z only? How can we obtain another equation from (2) and (3) containing x and z only?

Perform these two eliminations and find x and z from the resulting equations.

Now find y by substituting in any one of the given equations and verify by showing that the values you have found for x, y and z will satisfy all of the given equations.

The solution might be written in the following form:

Eliminate y from (1) and (2), $\therefore\ 11x+2z=57.$ (4)
" " " (2) " (3), $\therefore\ 7x-z=34.$ (5)
" z " (4) " (5), $\therefore\ x=5.$
Substitute $x=5$ in (4), $\therefore\ z=1.$
" $x=5$ and $z=1$ in (1), $\therefore\ y=2.$

$$\therefore\ x=5,\ y=2,\ z=1.$$

Of course it will be seen that any other unknown might have been eliminated twice from two pairs of the equations.

Thus we might have eliminated z from (1) and (2) and also from (1) and (3), and thus obtained two equations in x and y. We might then have completed the solution as before.

Solve the equations by this plan. Also solve them by two eliminations of x.

Which letter do you think is easiest to eliminate twice?

Note that the solution is completed only when the values of all of the unknowns have been found.

EXERCISE 77 (1-4, Oral)

$$\begin{aligned} x+y+z&=35 \quad (1)\\ x+y-z&=25 \quad (2)\\ x-y+z&=15 \quad (3)\end{aligned}$$

1. What operation will eliminate both x and y from (1) and (2)? What is the value of z? of $x+y$?

2. What operation will eliminate both y and z from (2) and (3)? What is the value of x? of $y-z$?

3. How can you eliminate both x and z from (1) and (3)? What is the value of y? of $x+z$?

4. In No. 5 below, which letter is simplest to eliminate from two pairs of the equations? Which in No. 6? Which in No. 7?

Solve and verify:

5. $$\begin{aligned} x+2y+3z&=16,\\ x+3y+4z&=24,\\ x+4y+10z&=41.\end{aligned}$$

6. $$\begin{aligned} 2x-y+3z&=7,\\ 3x+y-4z&=7,\\ 6x-y+5z&=21.\end{aligned}$$

7. $$\begin{aligned} 4x-3y+z&=10,\\ 6x-5y+2z&=17,\\ x+y+z&=8.\end{aligned}$$

8. $$\begin{aligned} x+y-z&=16,\\ x-y+z&=4,\\ x+y+2z&=22.\end{aligned}$$

9. $$\begin{aligned} x+2y+3z&=32,\\ 4x-5y+6z&=27,\\ 7x+8y-9z&=14.\end{aligned}$$

10. $$\begin{aligned} x+y&=25,\\ y+z&=75,\\ z+x&=70.\end{aligned}$$

11. $$\begin{aligned} x+2y&=12,\\ 3y+4z&=2,\\ 5z-2x&=-21.\end{aligned}$$

12. $$\begin{aligned} 3(z-1)&=2(y-1),\\ 4(y+x)&=9z-4,\\ 7(5x-3z)&=2y-9.\end{aligned}$$

13. $$\begin{aligned} 3+x&=5+4y,\\ z+x&=3y,\\ 7y&=z+2.\end{aligned}$$

14. $$\begin{aligned} \tfrac{1}{2}x+\tfrac{1}{3}y+\tfrac{1}{4}z&=36,\\ \tfrac{1}{5}x+\tfrac{1}{15}y+\tfrac{1}{20}z&=10,\\ \tfrac{1}{4}x+\tfrac{1}{8}y+\tfrac{1}{10}z&=43.\end{aligned}$$

15. $$\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=\frac{x}{3}+\frac{y}{4}+\frac{z}{5}=\frac{x}{4}+\frac{y}{5}+\frac{z}{6}=1.$$

16.* If $x+2y=25$, $y+3z=55$, $z+4x=35$, find the value of $x+y+z$.

17. If $x-y+z=9$, $2x+y=8$, $y-4z=5$ and $x+y+z+w=12$, find w.

18. If ax^2-bx+c is 8 when $x=1$, 8 when $x=2$, and 10 when $x=3$, find a, b and c.

19. If ax^2+bx+c is 9 when $x=1$, -3 when $x=-1$, 18 when $x=2$, find its value when $x=3$.

20. Determine three numbers whose sum is 9, such that the sum of the first, twice the second and three times the third is 22, and the sum of the first, four times the second and nine times the third is 58.

21. If $a+b=12$, $b+c=15$, $c+d=19$, find $a+d$.

111. Special Forms of Equations. Two equations of the first degree in x and y will usually determine the values of x and y.

Consider the following sets of equations:

(1) $2x-3y=10$, $4x+5y=42$. (2) $2x-3y=10$, $4x-6y=20$. (3) $2x-3y=10$, $4x-6y=30$.

In (1), if the two equations are solved in the usual way we find that $x=8$, $y=2$ will satisfy both of the equations, and no other values of x and y will satisfy them.

We therefore say that these equations are **determinate**, that is, they determine the values of x and y.

In this case the second equation can not be deduced from the first, nor the first from the second. We therefore say that the equations are **independent.**

In (2), the second equation may be deduced from the first by multiplying by 2. These equations are **dependent** and not independent as in (1).

Any number of values of x and y will satisfy both equations, because any values which will satisfy the first will also satisfy the second. These equations are therefore **indeterminate.**

In (3), if the first equation is true, the second can not be true. They are therefore said to be **inconsistent** or impossible, and no values of x and y can be found to satisfy both of them.

We thus see that *two equations in two unknowns can have a definite solution only when the equations are independent and consistent.*

In this set of equations, the third may be obtained by adding the other two. They are therefore dependent equations and consequently indeterminate.

$$\begin{aligned} 3x+2y-z&=5,\\ 4x-y+3z&=20,\\ 7x+y+2z&=25. \end{aligned}$$

EXERCISE 78

1. Find three pairs of values of x and y which satisfy the equation $2x-3y=12$.

2. Solve $2x+3y=13$, $5x-y=24$. Is it possible that $2x+3y=13$, $5x-y=24$ and $4x+5y=19$ can be true at the same time ?

3. What is peculiar about the equations $4x+y=17$, $8x+2y=35$? Also about $8x+12y=60$, $6x+9y=45$?

4. Show that the equations

$$x+z+4=3y, \quad 3x+z=2y+6, \quad 2x+y=10,$$

are indeterminate. If $z=5$, solve the equations.

5. Find two solutions of the simultaneous equations :

$$x+y+z=10, \quad 3x-2y-z=7.$$

For what values of a will the following sets of equations be consistent :

6.* $\begin{aligned} 3x-\ \ y&=5, \\ x+2y&=25, \\ x+4y&=a. \end{aligned}$

7. $\begin{aligned} 3x+2y&=7, \\ 10x-4y&=2, \\ 3x+ay&=11. \end{aligned}$

8. $\begin{aligned} 9x-ay&=0, \\ 3x-\ \ y&=2, \\ 5x-\tfrac{10}{3}y&=\tfrac{5}{3}. \end{aligned}$

9. Show that these equations are inconsistent :

$$2x+3y-3z=20, \quad 3x+7y-2z=5, \quad x+2y-z=6.$$

112. Special Fractional Equations.

Ex.—Solve

$$\frac{3}{x}-\frac{8}{y}=11, \qquad (1)$$

$$\frac{4}{x}+\frac{2}{y}=21. \qquad (2)$$

Here we could obtain the solution in the usual way by removing the fractional forms, by multiplying each equation by xy. See if you can complete the solution by this method.

It is simpler, however, to eliminate y from the equations as they stand.

Thus, multiplying (2) by 4 and adding

$$\frac{19}{x} = 95, \quad \therefore\ 95x = 19, \quad \therefore\ x = \frac{1}{5}.$$

Substitute $x = \frac{1}{5}$ in (1) and $15 - \frac{8}{y} = 11$.

$$\therefore\ \frac{8}{y} = 4, \qquad \therefore\ y = 2.$$

The solution, therefore, is $x = \frac{1}{5}$, $y = 2$.
Verify this result.

EXERCISE 79

Solve and verify:

1. $\frac{6}{x} + \frac{7}{y} = 2, \quad \frac{24}{x} - \frac{21}{y} = 1.$

2. $\frac{2}{x} + \frac{7}{y} = 29, \quad \frac{5}{x} - \frac{6}{y} = 2.$

3. $\frac{3}{x} + \frac{5}{y} = 19, \quad \frac{7}{x} - \frac{9}{y} = 3.$

4. $\frac{9}{x} - \frac{4}{y} = 8, \quad \frac{13}{x} + \frac{7}{y} = 101.$

5. $\frac{3}{x} + 2y = 15, \quad \frac{5}{x} - 3y = 6.$

6. $\frac{1}{x} + \frac{2}{y} = 11, \quad \frac{3}{y} - \frac{2}{z} = 2, \quad \frac{4}{z} - \frac{1}{x} = 17.$

7. $3y - 5x = xy, \quad 2y + 3x = 26xy.$

8. $\frac{3}{x} + \frac{1}{2y} = \frac{15}{2x} + \frac{1}{3y} = \frac{11}{6}.$

9. $\frac{5}{x} + \frac{3}{y} = \frac{135}{x} - \frac{75}{y} = 30.$

10. $3x + \frac{2}{y} - 1 = 12x + \frac{5}{y} + 14 = \frac{1}{y} - 2x - 14.$

11. $\frac{3}{x} + \frac{2}{y} - 4z = \frac{1}{x} + \frac{4}{y} = 12z - \frac{1}{x} + \frac{5}{y} = 8z + 17.$

113. Problems leading to Simultaneous Equations. In Chapter VII. we have had illustrations of problems which were solved by using equations of two unknowns. We now give some further examples on special subjects which were not then considered.

The number 47 might be written $4 \cdot 10+7$. What is the sum of the digits of this number? What number would be formed by reversing the digits? What is the sum of the number and the reversed number? What is the sum of the digits of the reversed number?

Ex. 1.—A number has two digits. If 18 is added to it the digits are reversed. The sum of the two numbers is 88. Find the number.

Let x = the units digit and y the tens digit,

$\therefore$ the number $=10y+x$,

and the reversed number $=10x+y$.

$$\therefore\ 10y+x+18=10x+y, \qquad (1)$$

and
$$10y+x+10x+y=88. \qquad (2)$$

Simplifying (1), $9x-9y=18$ or $x-y=2$,

" (2), $11x+11y=88$ or $x+y=8$.

Solving $x=5,\ y=3$.

$\therefore$ the required number is 35.

Verification: $35+18=53,\ 35+53=88$.

Ex. 2.—If 4 be added to the numerator of a fraction and 3 to the denominator, the fraction becomes $\frac{1}{2}$. If 2 had been subtracted from the numerator and 5 from the denominator the result would have been $\frac{1}{6}$. Find the fraction.

Let $\dfrac{x}{y}$ = the fraction,

$$\therefore\ \frac{x+4}{y+3}=\frac{1}{2} \quad \text{and} \quad \frac{x-2}{y-5}=\frac{1}{6}.$$

$$\therefore\ 2x+8=y+3 \quad \text{and} \quad 6x-12=y-5.$$

$$\therefore\ 2x-y=-5 \quad \text{and} \quad 6x-y=7.$$

Complete the solution and verify.

Sometimes the solution of a problem may be simplified by using some function of x instead of x to represent one of the unknowns.

Thus, if two numbers are in the ratio of 7 to 6, we might represent the larger number by x and then the smaller would be $\frac{6}{7}x$.

A better way, however, would be to represent the larger by $7x$, and then the smaller would be $6x$. By doing so we get rid of the use of fractions.

Ex. 8.—The incomes of A and B are in the ratio of 3 to 2, and their expenses in the ratio of 5 to 3. Each saves \$400 a year. Find their incomes and expenses.

Let $\$3x = A$'s income, then $\$2x = B$'s income.
Let $\$5y = A$'s expenses, then $\$3y = B$'s expenses.
$\therefore\ 3x - 5y = 400$ and $2x - 3y = 400$.
Solving $x = 800$ and $y = 400$.
$\therefore$ A's income $= \$3x = \$2400$ and B's $= \$1600$.
$\therefore$ A's expenses $= \$5y = \$2000$ and B's $= \$1200$.

Note.—In solving the problems in the exercise following, the pupil will find that he frequently has the choice of using one, two or more unknowns. Except in special cases, he is advised to use as small a number of unknowns as possible. In each case the results should be verified.

EXERCISE 30

1. If 10 men and 4 boys, or 7 men and 10 boys, earn \$96 in a day, find a man's wages per day.

2. Two numbers are in the ratio of 5 to 7 and their difference is 10. What are the numbers?

3. The sum of three numbers is 370. The sum of the first two is 70 more than the third and six times the first is equal to four times the third. Find the numbers.

4. Find three numbers such that the results of adding them two at a time are 29, 33, 36.

5. Divide 429 into three parts so that the quotient of the first by 7, the second by 4 and the third by 2 will all be equal.

6. A workman can save \$200 a year. He goes to another town where his wages are 10% greater and his expenses are 5% less, and he can now save \$310 a year. What are his wages now?

7. The denominator of a fraction exceeds the numerator by 3. If 2 is subtracted from each term, the fraction reduces to [illegible]. Find the fraction.

8. Divide 120 into three parts, so that $\frac{3}{4}$ of the first part is greater than the second by 5 and $\frac{3}{4}$ of the second part is greater than the third by 10.

9. If 6 men and 2 boys earn \$56 in 2 days and 7 men and 5 boys earn \$57 in $1\frac{1}{2}$ days, how long will it take 3 men and 4 boys to earn \$60 ?

10. A number between 10 and 100 is 8 times the sum of its digits, and if 45 be subtracted from it, the digits are reversed. Find the number.

11. The difference of the two digits of a number is 4. The sum of the number and the reversed number is 110. Find the number.

12. The sum of the two digits of a number is 14, and when 18 is added to the number the digits are reversed. Find the number.

13. When 1 is added to both terms of a fraction the result is $\frac{1}{3}$. If 9 had been subtracted from the denominator only the result would have been $\frac{1}{2}$. Find the fraction.

14. A number consists of two digits whose sum is 11. If the order of the digits be reversed, the number thus obtained is greater by 7 than twice the original number. What is the number ?

15. The difference between the digits of a number less than 100 is 6. Show that the difference between the number and the number formed by reversing the digits is always 54.

16. The sum of the reciprocals of two numbers is $\frac{3}{10}$. Six times the reciprocal of the first is greater than five times the reciprocal of the second by $\frac{1}{3}$. Find the numbers. (The reciprocal of x is $\frac{1}{x}$.)

17. Divide 150 into two parts such that the quotient obtained by dividing the greater by the less is 3 and the remainder is 2.

18. I wish to obtain 100 lb. of tea worth 34c. per lb. by mixing tea worth 30c. per lb. with tea worth 40c. per lb. How much of each must I take ?

19. Three pounds of tea and 10 of sugar cost \$2·40. If tea is increased 10% in price and sugar decreased 10%, they would cost \$2·52. Find the price of each per lb.

20. Two numbers are in the ratio of 7 to 5. What quotient is obtained when three times their sum is divided by six times their difference ?

21. Show that the sum of any number of two digits and the number formed by reversing the digits is always divisible by 11 and that the difference is divisible by 9.

22. A number has three digits, the middle one being 0. If 396 be added the digits are reversed. The difference between the number and five times the sum of the digits is 257. What is the number ?

23. Divide 126 into four parts, so that if 2 be added to the first, 2 be subtracted from the second, the third be multiplied by 2, and the fourth be divided by 2, the results will all be equal. (Let the result $=x$.)

24. There are three numbers such that when each is added to twice the sum of the remaining two the results are 44, 42, 39. Find the numbers.

25. The sum of the three digits of a number is 12. If the units and tens digits be interchanged the number is increased by 36, and if the hundreds and units be interchanged it is increased by 198. Find the number.

26. Find three numbers such that the first with $\frac{1}{3}$ of the sum of the other two, the second with $\frac{1}{4}$ of the others, and the third with $\frac{1}{5}$ of the others, shall each be 25.

27. A piece of work can be done by *A* working 6 days and *B* 21 days, or by *A* working 8 days and *B* 18 days. In what time could each of them complete it alone ?

28. Divide 84 into four parts, so that the first is to the second as 2 to 3, the second to the third as 3 to 4, and the third to the fourth as 4 to 5.

29. Of what three numbers is it true that the sum of the reciprocals of the first and second is $\frac{1}{2}$, of the first and third is $\frac{1}{3}$ and of the second and third is $\frac{1}{4}$?

30. Two numbers consist of the same three digits but in inverted order. The sum of the numbers is 1029. The sum of the digits of each is 15 and the difference of the units digits is 5. Find the numbers.

31. A stream flows at 2 miles per hour. A man rows a certain distance up stream in 5 hours and returns in $1\frac{2}{3}$ hours. How many miles per hour could he row in still water ?

32. A rancher sold 50 head of horses, part at $125 a head and the balance at $150 a head. After spending $50 he was able to make the first payment of $\frac{1}{3}$ of the purchase price of 1200 acres of land at $18 per acre. How many horses did he sell at $125 a head ?

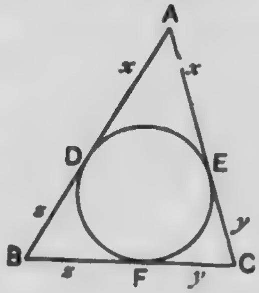
33. A number consists of a units digit and a tenths digit, the units digit being the greater by 1. The sum of the digits is less than twice the number by 2. Find the number.

34. A grocer spent \$120 in buying tea at 60c. a lb., and 100 lb. of coffee. He sold the tea at an advance of 25% on cost and the coffee at an advance of 20%. The total selling price was \$148. Find the number of lb. of tea purchased.

35. When 2 is subtracted from each term of a fraction the result is equal to ½. Show that the result would have been the same if 1 had been subtracted from the numerator only.

36. It is shown in geometry that the two tangents drawn to a circle from a point are equal. Thus, in the figure $AD = AE$, etc. If $AB = 15$, $BC = 14$, $CA = 13$, find x, y and z.

37. If the sides of a triangle are 10, 15 and 19, where will the inscribed circle touch the sides? (See figure of preceding example.)



38. If A can do a piece of work in m days and B can do it in n days, in what time can they do it working together?

If x is the number of days required,

$$\text{then } \frac{1}{x} = \frac{1}{m} + \frac{1}{n}. \quad \therefore \ x = \frac{mn}{m+n}.$$

39. Use the preceding result to find in what time A and B working together can do a piece of work which could be done by A and B seperately in the following number of days:

(1) A in 10, B in 15. (2) A in 20, B in 5. (3) A in ¾, B in 1½.

EXERCISE 81 (Review of Chapter XII)

Solve and verify:

1. $7x - 8y = 10,$
   $3x - 2y = 10.$

2. $4x + 7y = -1,$
   $3x - y = 3.$

3. $73x + y = 75,$
   $x + 73y = 147.$

4. $\frac{10}{x} - \frac{12}{y} = 14,$
   $\frac{7}{x} + \frac{4}{y} = 16.$

5. $\frac{7}{x} - \frac{5}{y} = 3,$
   $\frac{2}{x} + \frac{25}{2y} = 12.$

6. $12y - 8x = 2xy,$
   $3y + 4x = 2xy.$

7. $\dfrac{x+1}{y}=7,\ \dfrac{x}{1+y}=6.$

8. $\dfrac{x+1}{y+2}=\dfrac{x+3}{2y+1}=2.$

9. $\dfrac{x-y}{3}=\dfrac{2x+3y}{5}=-4.$

10. $\dfrac{x-2y}{3}=\dfrac{3x-y}{7},$
$3(x+y)=8.$

11. $3x-\dfrac{y-3}{5}=6,$
$4y+\dfrac{x-2}{3}=12.$

12. $\dfrac{x-1}{2}=\dfrac{y-3}{4}=\dfrac{z-5}{6},$
$x+y+z=33.$

13. $x+y+z=-3,$
$x+2y+z=0,$
$3x+y+6z=0.$

14. $2x+3y-z=5,$
$3x-4y+2z=1,$
$4x-6y+5z=7.$

15. $\dfrac{x}{3}=\dfrac{y}{8}+1=\dfrac{z}{2}-3.$
$x+y+z=24.$

16. $\dfrac{x-2}{5}-\dfrac{10-x}{3}=\dfrac{y-10}{4},$
$\dfrac{2y+4}{3}-\dfrac{2x+y}{8}=\dfrac{x+13}{4}$

17 $\dfrac{x-1}{3}-\dfrac{y+5}{12}=\dfrac{x+2}{60},$
$(x-1\tfrac{1}{2})(y-1\tfrac{1}{3})=xy-5.$

18. $\tfrac{1}{2}x-\tfrac{3}{4}y+z+1=3(x-y)+5z+4=x+6y-2z-9=0.$

19. $\dfrac{5x-3y}{4}=4y-2z=\dfrac{10z+\frac{1}{3}x}{2}=6.$

20. $x+y=5,\ y+z=3,\ z+w=1,\ x-w=3.$

21. $31x+28y=146,\quad 28x+31y=149.$ (Add and subtract th equations and remove common factors.)

22. $97x-59y=329,\ 59x-97y=139.$

23. What values of x and y will make $\dfrac{x-2y}{3}$ and $\dfrac{x+y}{5}$ each equal to $x-10$?

24. Show that $x+y+z=12,\ 3x+4y-5z=-22,\ 10x+12y-6z=4,$ are indeterminate.

25. Divide unity into two parts so that 18 times the first part may exceed 12 times the second by 13.

26. A number of two digits is four times the sum of its digits, and if 18 be added to the number the digits are reversed. What is the number?

27. The tens digit of a number is twice the units digit. When the number is divided by the sum of the digits what must the quotient be?

28. F[illegible] a fraction equal to [illegible] such that [illegible] of the denominator exceeds [illegible] of the numerator by 8.

29. Two persons who are 30 miles apart are together after 5 hours if they walk in opposite directions, but are not together for 15 hours if they walk in the same direction. What are their rates ?

30. *A*'s age is equal to the combined ages of *B* and *C*. Ten years ago *A* was twice as old as *B*. Show that 10 years hence *A* will be twice as old as *C*.

31. A bill of \$19·50 was paid in half-dollars and quarters and four times the number of quarters exceeded twice the number of half-dollars by 12. How many of each were used ?

32. If 5 lb. of tea and 8 lb. of coffee cost \$5·80, and coffee advances 10% in price and tea 15% and they now cost \$6·53, find the prices per lb. of each before the advance.

33. I invest a certain sum at 4% and another sum at 6% and receive \$42 interest. If the sums had been interchanged I would have received \$8·50 more. What were the sums ?

34. If each side of a rectangle is increased by 5 feet the area is increased by 275 square feet. If each side is decreased by 5 feet the area is decreased by 225 square feet. Show that the sides can not be determined from these conditions.

35. Solve $\frac{7x+3y-z}{10}=\frac{7x+y}{9}=4x-z=1$.

36. A grocer wishes to mix tea worth 30c. a lb. with tea worth 40c. to make a mixture weighing 60lb. worth 36c. a lb. How many lb. of each must he use ?

37. If $3x^2-2x+5=ax^2+bx+c$, when $x=1$ or $x=2$ or $x=3$, show that $a=3$, $b=-2$, $c=5$.

38. The tens digit of a number exceeds the units digit by 3. By how much is the number decreased by inverting the digits ?

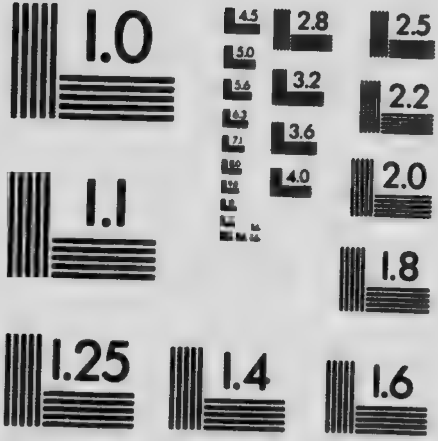
39. A train is 27 minutes late when it makes its usual trip at 28 miles per hour and is 42 minutes late when it runs at 27 miles per hour. What is the distance ?

40. A piece of work can be done by *A* working 6 days and *B* 16 days or by *A* working 9 days and *B* 14 days. How long would it take each alone to do it ?

41. A number has three digits, the units being [illegible] of the tens and [illegible] of the hundreds. If 396 be subtracted the digits a[illegible]versed. Find the number.

MICROCOPY RESOLUTION TEST CHART

(ANSI and ISO TEST CHART No. 2)



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42. When the greater of two numbers is divided by the less the quotient is 5 and the remainder is 2. When 12 times the less is divided by the greater the quotient is 2 and the remainder is 12. Find the numbers.

43. Find four numbers such that when each is added to twice the sum of the remaining three, the results are 46, 43, 41 and 38 respectively.

44. If the sum of two numbers is a times the greater and the difference is b times the smaller, show that $a-b+ab=2$.

CHAPTER XIII

GEOMETRICAL REPRESENTATION OF NUMBER

114. Function of x. The value of the expression $3x-2$ depends upon the value of x.

Thus, when $x = 4, 3, 2, 1, 0, -1, -2, -3, -4,$
$3x-2 = 10, 7, 4, 1, -2, -5, -8, -11, -14.$

When the value of an expression depends upon the value of x, the expression is called a **function of x.**

Thus, $2x-3$, $5x$, $\frac{1}{2}x+1$, are functions of x.
What is the value of each of these functions when $x=2, 1, 0, -1, -2$?

Instead of repeating the words "the expression" or "the function," we might represent the function by a symbol, say y.

Thus, if $y=5x+1$, when $x=1$, $y=6$; $x=3$, $y=16$.
If $y=\frac{1}{2}x+4$, what are the values of y when x has the values 6, 3, 0, -1, -8?

115. Variables and Constants. A quantity that has not always the same value is called a **variable,** while a quantity whose value does not change is called a **constant.**

Thus, the population of a city and the height of the barometer are variables, while the number of days in a week and the length denoted by an inch are constants.

NOTE.—To do the work of this chapter properly, pupils should be supplied with squared paper. Paper ruled in tenths or eighths of an inch will be found most satisfactory.

116. Connected Variables. Two variables may be so connected that for every change in the value of one there is a corresponding change in the value of the other.

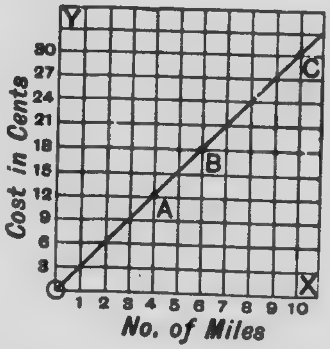
Thus, if $y=2x+5$, for each value of x there is a corresponding value of y. Here x and y are variable quantities, but 5 is a constant.

In arts. 20 and 21 we have shown how the changes in two variable quantities may be represented by a diagram. Those diagrams show that for each variation in time there is a corresponding variation in **temperature.**

117. Graph. A line so drawn as to exhibit the nature of the relation of two variables is called a **graph.**

118. Arithmetical Graphs. The solution of many problems in arithmetic might be represented graphically as follows:

Ex. 1.—The passenger rate on a railway is 3 cents per mile. Represent graphically the amount charged for any number of miles from 1 to 10.



In the diagram each unit on the horizontal line OX represents 1 mile and each unit on the vertical line OY represents 3 cents.

The point A shows that the cost for 4 miles is 12 cents. What does the point B show? The point C?

Read from the figure the cost for 2 miles, 5 miles, 9 miles. How far can I travel for 9 cents, 21 cents, 27 cents?

Ex. 2.—Represent graphically the simple interest at 2% on $100 for any number of years from 1 to 6.

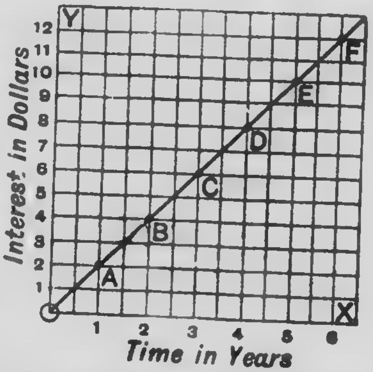
Reading from the diagram (on the next page), what is the interest on $100 in 2 years? In 5 years? In 4½ years?

What does the point A show? The point C? The point D? The point half way between C and D?

In how many years is the interest $8, $6, $5, $11, $6·50?

Place a ruler on the points marked *A, B, C, D, E, F.* What peculiarity do you notice ?

Make a similar diagram, on squared paper, which will give the interest on $200 at 4% for any number of years from 1 to 7. So that your diagram will not occupy too much space vertically, suppose each unit on *OY* to represent $4 instead of $1.

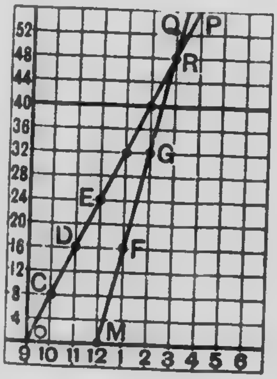


Read from your diagram the interest for 3 years, for 5 years. In how many years is the interest $8, $24, $4, $12, $44 ?

Ex. 3.—A starts at 9 A.M. on a bicycle at 8 miles per hour. He is followed at noon by B on an automobile at 16 miles per hour. When and where will B overtake A ?

Each space on the horizontal line represents 1 hour, and on the vertical line 4 miles.

At the end of successive hours A's position will be *C, D*, etc., and B's will be *F, G*, etc.



The line *OP* is the graph of A's journey and *MQ* is the graph of B's.

The diagram shows that B overtakes A at the point *R*, which is 48 miles from the starting point and that the time is 3 P.M.

How far is A ahead at 12 o'clock ? at 1 ? at 2 ?

Solve the problem otherwise and compare the results.

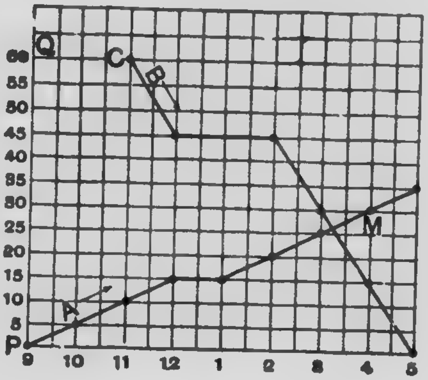
Ex. 4.—A starts from *P* at 9 A.M. to go to *Q*, a distance of 60 miles, travelling at 5 miles per hour. He stops at 12 for one hour for lunch. B starts from *Q* at 11 A.M. to go to *P*, travelling at 15 miles per hour. An accident detains him from 12 to 2. Where and when will they meet ?

The graph of A's journey is represented by the upward line drawn from *P*, and of B's by the downward line drawn from *C*.

The position of each at the end of the successive hours is marked on the diagram. (See next page.)

They meet at *M* at about 3.15 P.M. and at a distance from *P* of about 26 miles.

How far are they apart at the end of each hour from 11 to 5? When did B reach *P*?



We might solve the problem algebraically.

Suppose they are together x hours after 9 A.M.

Then A has travelled $x-1$ hours at 5 miles per hour, and B $x-4$ hours at 15 miles per hour,

$$\therefore\ 5(x-1)+15(x-4)=60.$$

Solve and compare with the results obtained graphically.

119. Graphical Results only Approximate. The last problem illustrates the fact that the results obtained by graphical methods are approximate only. When the problem is solved algebraically we find that they will meet 26¼ miles from *P* at 3.15 P.M.

EXERCISE 82

1. A man walks at the rate of 4 miles per hour. Construct a graph to show the distance he walks in any number of hours from 1 to 10. Read from the graph the distance walked in 3, 5, 7½, 9½ hours. (Take two units on the horizontal line to represent 1 hour and one unit on the vertical line to represent 2 miles.)

2. In Ex. 1, if he rests 30 minutes after walking each 4 miles, how long will it take him to walk 8, 12, 14, 7, 17 miles? How far will he have gone in 1½, 3½, 5½, 8½ hours?

3. *A* starts running at the rate of 6 yards per second, and 4 seconds later *B* starts from the same place at 9 yards per second. Construct a graph to show when and where *B* will overtake *A*. How far apart are they 6 seconds after *A* started? When was *A* 12 yards ahead of *B*?

4. Oranges sell at 19 cents per dozen. Make a graph from which you can read off the price to the nearest cent of any number from 1 to 12. What is the cost of 2, 5, 7, 8, 10 oranges? How many can I buy for 3, 5, 8, 13, 16 cents?

5. If 8 kilometres equal 5 miles, construct a graph which will enable you to change into miles any number of kilometres up to 20. Read the approximate number of miles in 3, 5, 11, 13, 16, 19, 20 kilometres.

6.* A starts from Toronto at 12 miles per hour to motor to Hamilton, a distance of 40 miles. An hour and a half later B starts from Hamilton to drive to Toronto at 8 miles per hour. By means of a graph, find when and where they will meet.

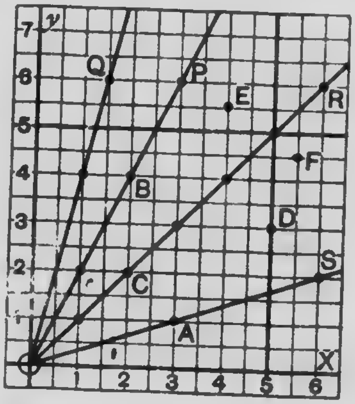
7. The distance from *A* to *B* is 10 miles, *B* to *C* 8, *C* to *D* 8, *D* to *E* 10 miles. A mail train, which leaves *A* at 10 A.M., arrives at *B* at 10.24, *C* at 10.48, *D* at 11.12, *E* at 11.40. An express train leaves *E* at 10.24 and without stopping reaches *A* at 11.28. If the mail train stops 4 minutes at each station, show graphically:

(*a*) when and at what point they pass each other,

(*b*) how far they are apart at 10.30 and at 11.12,

(*c*) when the express passes through *B*.

120. The Axes. In the diagram the line *OX* is called the **axis of** x and *OY* the **axis of** y. We will call the measurement along *OX*, x, and along *OY*, y.



For the point *A* the x measurement is 3, and the y measurement is 1. What are the x and y measurements for the points *B*, *C*, *D*, *E*?

Examine the x and y measurements for each point marked on the line *OP*. What equation connects the values of x and y for each point on the line *OP*? For each point on *OQ*, *OR*, *OS*?

OP is the graph of the equation $y=2x$, *OQ* of $y=4x$, *OR* of $y=x$, and *OS* of $y=\frac{1}{3}x$.

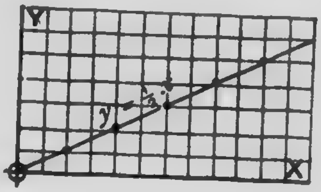
The x and y measurements of every point on the line *OP* satisfy the equation $y=2x$. This equation is not satisfied by the x and y of any point not on the line OP. Is it satisfied by the x and y of the points A, C, D, E?

121. Equation of a Line. Since the equation $y=2x$ is satisfied by the values of x and y for each point on the line OP and by no other points, the equation $y=2x$ is called **the equation of the line OP.**

What is the equation of OQ? of OR? of OS?

122. To Construct the Graph of a given Equation.

Ex.—Construct the graph of $y=\frac{1}{2}x$.



Here when $x=2$, $y=1$,
$x=4$, $y=2$,
$x=6$, $y=3$, etc.

To find the point where $x=2$, $y=1$, count 2 units from O along OX and then 1 unit upwards. Find in a similar manner the points where $x=4$, $y=2$; $x=6$, $y=3$; $x=8$, $y=4$; $x=10$, $y=5$.

Join all the points located. They are all seen to lie on the same straight line passing through O.

This line is the graph of the equation $y=\frac{1}{2}x$.

123. The Origin. All the lines we have so far considered have been drawn through the point O. This point is called **the origin.**

The x and y measurements of the origin are $x=0$, $y=0$. These values satisfy the equation $y=\frac{1}{2}x$, and consequently the graph of this equation should pass through the origin as the figure shows.

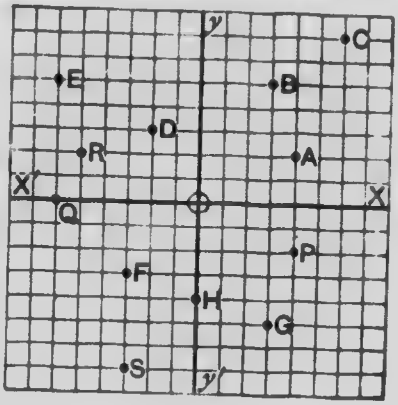
EXERCISE 33

Construct the graphs of:

1. $y=x$.	2. $y=5x$.	3. $y=\frac{1}{4}x$.
4. $y=\frac{2}{3}x$.	5. $4y=x$.	6. $5y=2x$.

124. Coordinates. In the diagram O is the origin, XOX' is the axis of x and YOY' is the axis of y. The x measurement of the point A is 4, and the y measurement is 2.

These are called the **coordinates** of the point A, 4 being the **x coordinate** and 2 the **y coordinate.** The coordinates of the point A are written (4, 2), the x coordinate being written first.



Similarly, the coordinates of B are (3, 5) and of C (6, 7).

So far all our measurements have been made from O towards the *right* and then *upwards.* We might also measure from O to the *left* and *downwards.* When we do so we indicate the change in direction by a change in algebraic sign.

Thus, to reach the point D we measure 2 units to the left and then 3 units upwards. Therefore the coordinates of D are $(-2, 3)$.

Similarly, the coordinates of E are $(-6, 5)$, of F $(-3, -3)$, of G $(3, -5)$, and of H $(0, -4)$.

What are the coordinates of P, Q, R, S, O ?

Mark on the diagram the points whose coordinates are $(2, 2)$, $(-4, 6)$, $(2, -5)$, $(-1, -3)$, $(0, 4)$, $(-3, 0)$, $(0, -3)$.

Using squared paper, take the origin at the intersection of two lines. Mark the points which would be indicated thus : $(2, -3)$, $(5, 6)$, $(-3, 7)$, $(-6, -2)$, $(3, 0)$, $(0, 3)$. Mark any other four points and show how their positions would be indicated.

125. Quadrants. That part of the plane between OX and OY is called the first quadrant, between OY and OX' the second quadrant, between OX' and OY' the third and between OY' and OX the fourth.

Thus, the points A and B are in the first quadrant, D and E in the second, F and S in the third and P and G in the fourth.

In which quadrant are both x and y negative ?

126. Plotting Points. When we represent the position of a point with respect to the axes XOX' and YOY', we are said to **plot the point.**

When two points are plotted the distance between them may be obtained by adjusting the points of the compasses to the two points and transferring the compasses to the line OX, or any other line, and reading off the distance.

Plot the points (3, 5) and (6, 1), and see if the distance between them is 5.

EXERCISE 54

1. In what quadrants are the points (3, 4), (4, -1), (-5, 3), (-1, -2) ?

2. Plot the points (1, 2), (4, -6), (-3, 7), (-5, -2).

3. Plot the points (5, 0) and (-3, 0). What is the distance between them ?

4. Where are the points (0, 0), (0, 2), (-5, 0), (4, 0) situated ?

5.* What is the distance between the points (6, 4), (1, -8) ?

6. What kind of figure is formed by joining the points (0, 0), (4, 0), (4, 4), (0, 4) in order ? What is its area ?

7. What kind of a triangle is formed by joining the points (0, 2), (2, 6), (2, 2) ? What is its area ?

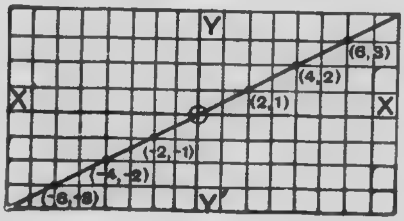
8. Plot the points (1, 1), (1, 3), (2, 1), (3, 3), (3, 1). Join them in order. What letter is formed ?

9. What is the area of the figure formed by joining (1, -3), (-5, -3), (-5, 6), (1, 6) in order ?

10. The angular points of a triangle are (6, 0), (3, 4), (-2, 0). Construct the triangle and find its area. Measure or calculate the lengths of the sides.

11. What is the length of the perpendicular from the point (5, 8) to the line joining (3, 2) and (7, 2) ?

127. Complete Graphs. In art. 122 we constructed the graph of the equation $y=\frac{1}{2}x$, but only for positive values of x and y.



The diagram, which is here repeated, shows that the line also passes through the points

(−2,−1), (−4,−2) and (−6,−3). This is as we would expect because

$$x=-2,\ y=-1\ ;\ \ x=-4,\ y=-2\ ;\ \ x=-6,\ y=-3,$$

all satisfy the equation $y=\frac{1}{2}x$.

128. Linear Equation. It is seen that the graphs of all the equations so far constructed have been straight lines. This is true concerning all equations of the first degree. For this reason an equation of the first degree is sometimes called a **linear equation.**

Since a straight line is fixed or determined when any two points on it are fixed, it follows that *to construct the graph of an equation of the first degree, we need to determine only two points on it.*

129. Lines not passing through the Origin. Every equation of the form $y=mx$ represents a straight line passing through the origin, because the equation is satisfied by $x=0$, $y=0$.

If the equation contains a term independent of x and y, it represents a straight line which does not pass through the origin.

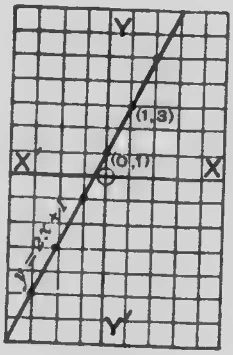
Thus, $y=2x+1$ represents a straight line which does not pass through the origin, because this equation is not satisfied by $x=0$, $y=0$.

Ex. 1.—Construct the graph of $y=2x+1$.

The coordinates of two points on the line are $x=0$, $y=1$ and $x=1$, $y=3$.

Locate these two points and draw the unlimited straight line which joins them. This is the required graph.

The diagram shows that it also passes through the points (2, 5), (−1, −1), (−2, −3), (−3, −5). Do the coordinates of these points satisfy the equation ?



In constructing the graph of an equation by locating two points on it, the pupil should try and determine two points whose coordinates are integers.

Ex. 2.—Construct the graph of $3x+4y=15$.

Here $y=\frac{15-3x}{4}$ and when $\begin{pmatrix} x=1, & y=3 \\ x=5, & y=0 \end{pmatrix}$.

Plot the points (1, 3) and (5, 0) and join them giving the required graph.

We might have found the points at which the graph cuts the axes. Thus, when $x=0$, $y=3\frac{3}{4}$ and when $y=0$, $x=5$. The required line is then found by joining the points $(0, 3\frac{3}{4})$ and (5, 0).

If the latter method is followed and fractions appear in the coordinates of either of the points found, the unit of measurement should be changed, in this case, by taking *four* spaces as the unit instead of *one*.

When the unit is not *one space*, it should be clearly shown on the diagram what the selected unit is

EXERCISE 86

1. Find two pairs of values of x and y which satisfy $x+y=6$. Plot the points whose coordinates are the values found and construct the graph of the equation $x+y=6$.

2. What are the coordinates of the points at which the graph in Ex. 1 cuts the axis of x, the axis of y?

Construct the graphs of the following equations:

3. $y=x+3$.
4. $y=x-3$.
5. $y=2x-3$.
6. $y=3x-2$.
7. $x+2y=7$.
8. $x-2y=7$.
9. $2x+3y=12$.
10. $3x-4y=16$.
11. $5x+6y=17$.

12. Construct the graph which cuts off 4 units from the axis of x and 6 units from the axis of y. Find the area of the triangle which this line forms with the axes.

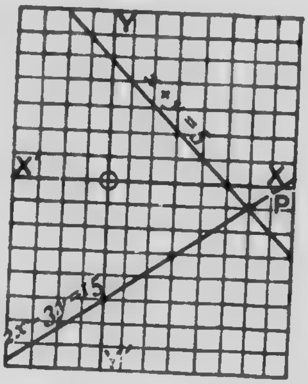
13. On the same sheet construct the graphs of $x-y=10$ and $x+2y=7$. What are the coordinates of the point at which they intersect? Do the coordinates of this point satisfy both equations?

14. Will the point (3, 4) lie on the graph of the equation $4x+3y=24$? Which of the following points lie on it: (2, 6), (0, 8), (6, 0), (9, −4), (5, 2), (−1, 9)? Verify by constructing the graph.

15. By constructing the graph of $2x+3y=24$, find three sets of positive integral values of x and y which satisfy this equation.

16. Why is there an unlimited number of positive integral values of x and y which will satisfy $2x-3y=24$, but only a limited number which will satisfy $2x+3y=24$?

130. Graphical Solution of Simultaneous Equations. In this diagram are shown the graphs of the equations $x+y=5$ and $2x-3y=15$.



The coordinates of the point P, at which the lines intersect, must satisfy both equations.

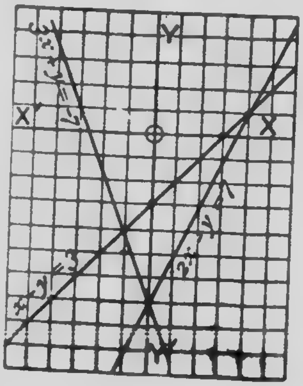
The coordinates of P are $(6,-1)$.

$\therefore$ $x=6$, $y=-1$, must be the values of x and y which satisfy both equations. We have therefore obtained the solution of these two equations graphically.

Since it is evident that two straight lines can intersect at only one point, it must follow that *there is only one pair of roots of two simultaneous equations of the first degree.*

In this diagram are shown the graphs of

$$(a)\ x-y=3,\quad (b)\ 2x-y=7,\quad (c)\ 3x+y=-7$$



At what point do the graphs of (a) and (b) intersect? (a) and (c)? (b) and (c)?

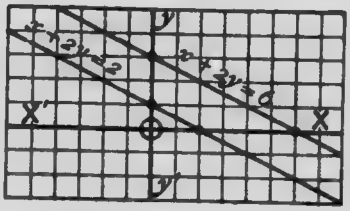
Is there any point which is common to the three lines? Are there any values of x and y which will satisfy these three equations at the same time?

When three equations in x and y are all satisfied by the same values of x and y, what peculiarity will appear in their graphs?

131. Special Forms of Equations.

(1) In this diagram are drawn the graphs of

$$x+2y=6 \text{ and } x+2y=2.$$



The lines which these equations represent are seen to be *parallel*, that is, there is no point at which they intersect. This is equivalent to saying that these equations have no solution.

Compare with art. 111, where inconsistent equations were discussed.

(2) If we draw the graphs of $x-3y=10$ and $2x-6y=20$ on the same sheet, we shall find that they represent the same straight line, so that any points which lie on the graph of one of them will also lie on the graph of the other.

These equations are indeterminate (art. 111).

(3) An equation like $x=3$ may be written $x+0y=3$.

This equation is satisfied by $x=3$ and any value of y.

Thus, $x=3$, $y=1$; $x=3$, $y=2$; $x=3$, $y=10$, etc., satisfy the equation. If we plot the points which have these coordinates, we see that $x=3$ represents a straight line parallel to the axis of y and at a distance 3 to the *right* of it.

Similarly, $x=-3$ represents a line parallel to the axis of y and at a distance 3 to the *left* of it. Also $y=-4$ represents a line parallel to the axis of x and at a distance 4 *below* it.

Thus *an equation which contains only one of the variables* x *and* y *represents a line parallel to one of the axes.*

What line does $x=0$ represent? $y=0$?

EXERCISE 86 (1-4, Oral)

1. What is the graph of $x=3$? Of $x+3=0$?

2. What is the graph of $y-4=0$? Of $2y+3=0$?

3. If $x+3y=11$, express x as a function of y and y as a function of x.

4. If $3x-2y=6$, express x as a function of y and y as a function of x.

Solve graphically and verify:

5. $x+3y=9$, $2x+y=8$.

6. $x+y=8$, $3x-4y=10$.

7. $x-2y=6$, $2x-3y=11$.

8. $2x+3y=6$, $3x+2y=14$.

9. $2y=x$, $10y=4x-2$.

10. $x+4y=9$, $3x-8y=-3$.

11. Show by graphs that the equations $x+y=5$, $2x+3y=12$, $3x-2y=5$ have a common pair of roots and find them.

12. Show graphically that $2x+3y=13$ and $\frac{1}{2}x+\frac{3}{4}y=2$ are inconsistent. What is peculiar about the graphs of these equations?

13. Show graphically that no values of x and y will satisfy all of the equations $x+y=4$, $2x-y=11$, $4x+y=13$. What values satisfy the first and second, the first and third, the second and third?

14. Show the coordinates of the points where the graph of $y=2x+3$ cuts (1) the axis of x, (2) the axis of y, (3) the graph of $y=6-x$.

15. Show by graphs that the values of x and y which satisfy $2x-3y+1=0$ and $5x-2y-14=0$ will also satisfy $3x-4y=0$ and $x-2y+2=0$.

EXERCISE 87 (Review of Chapter XIII)

1. At what point does the graph of $x+y=5$ cut the axis of x? The axis of y? Construct the graph. In the same way construct the graph of $x+4y=-4$. At what point do they intersect?

2. How does it appear geometrically that two equations of the first degree can have only one set of roots?

3.* Plot the points (0, 0), (−3, 4), (3, 12), (−2, 0). What is the distance between each consecutive pair of these points?

4. From a certain point a man walks 5 miles E., then 4N., then 2W., then 3N., then 3E., then 4S. Using squared paper, determin by measurement how far he is now from the starting point.

5. A man walks 8 miles W. and then 6S. Find by calculation how far he must now walk to reach a point 4 miles E. of his starting point.

6. If 11 lb. equal 5 kilogrammes, make a graph from which you can express any number of kilogrammes in lb. or lb. in kilogrammes. Read from the graph $3\frac{1}{2}$ kilogrammes in lb. and $8\frac{1}{2}$ lb. in kilogrammes.

7. What is the perimeter and area of the triangle whose angular points are (0, 0), (5, 0), (0, 12) ?

8. How do you show that the point (3, −2) lies on the graph of $5x-2y=19$? Which of the following lie on it : (6, 5), (1, −7), (- 3, −17), (4, 1), (−2, −12), (5, 3) ?

9. Find the area of the triangle formed by joining the points (5, 9), (8, −6), (−7, −6).

10. Draw the triangle whose vertices are (2, 0), (10, 0), (5, 6) and find its area. Why do the points (2, 0), (10, 0), (8, 6) determine a triangle of the same area ?

Solve graphically and verify :

11. $x+2y=12,\quad x-3y=2.$

12. $3x-4y=0,\quad 4x-3y=14.$

13. $y-x=4,\quad x=2.$

14. $y-2x=-3,\quad x+2y=14.$

15. $2x+7y=52,\quad 3x-5y=16.$

16. $y=\frac{1}{2}x+4,\quad y=\frac{1}{2}x+5.$

17. What is the area of the figure formed by the lines whose equations are : $x=4$, $x=-2$, $y=3$, $y=-1$?

18. What are the coordinates of the middle point of the line joining the points (2, 3) and (6, 5) ?

19. On the same sheet draw the graphs of the equations

$$y=x+4,\ y=4x-2,\ y=2x+2.$$

What peculiarity is presented by the graphs ? What conclusion do you draw concerning these equations ?

20. Draw the graphs of $2x+3y=20$, $4x+6y=35$ on the same sheet. What do you conclude as to the solution of these equations ?

Determine graphically whether these sets of equations are consistent or inconsistent :

21. $x-y=4,\quad 4x+y=26,\quad 2x-5y=2.$

22. $x+2y=10,\quad 3x-y=9,\quad 2x-y=1.$

23. $2x+y=8,\quad 3x+2y=13,\quad 5x-3y=9.$

24. Describe the triangle whose sides are represented by the equations : $3x+2y=14$, $5x-6y=-14$, $x+10y=-14$. What are the coordinates of its vertices ? (Verify by solving the equations in pairs.)

25. At what point do the graphs of $2x+3y=12$, $3x-2y=5$ intersect? At what angle do they seem to intersect?

26. A teacher's salary is increased by $50 each year. His salary for the first year is $750. Construct a graph from which you can read off his salary for any year. What is his salary for the 8th year? In what year would his salary be $1300?

27. In the process of solving $2x-3y=1$, $3x+2y=8$, by eliminating y we have

$$\begin{array}{l|l|r|r} 2x-3y=1, & 4x-6y=\ 2, & x=2, & x=2, \\ 3x+2y=8. & 9x+6y=24. & 2x-3y=1. & y=1. \end{array}$$

On the same sheet show the graphs of each of these sets of equations, and thus show that they all determine the same point and that the four sets are therefore equivalent

CHAPTER XIV

HIGHEST COMMON FACTOR AND LOWEST COMMON MULTIPLE

132. In Chapter IX. we defined the terms highest common factor and lowest common multiple, and showed how they were found in simple cases.

When the expressions under consideration can be factored, the H.C.F. and L.C.M. can at once be written down from the factored results.

A few examples are here given of a more difficult character than those previously considered.

Ex. 1.—Find the H.C.F. and L.C.M. of

$$x^2y+7xy^2+12y^3 \text{ and } x^3y-x^2y^2-12xy^3.$$

$$x^2y+7xy^2+12y^3 = y(x^2+7xy+12y^2) = y(x+4y)(x+3y).$$
$$x^3y-x^2y^2-12xy^3 = xy(x^2-xy-12y^2) = xy(x-4y)(x+3y).$$

Here the common factors are y and $x+3y$, and since the H.C.F. is the product of all the common factors,

$$\therefore \text{ the H.C.F.} = y(x+3y).$$

The L.C.M. is the expression with the lowest number of factors which will include all the factors of each expression,

$$\therefore \text{ the L.C.M.} = xy(x+4y)(x+3y)(x-4y).$$

Ex. 2.—Find the L.C.M. of

$$x^2-1,\ x^3+1,\ x^4-x \text{ and } x^4+x^2+1.$$

$$x^2-1=(x+1)(x-1).$$
$$x^3+1=(x+1)(x^2-x+1).$$
$$x^4-x=x(x^3-1)=x(x-1)(x^2+x+1).$$
$$x^4+x^2+1=(x^2+1)^2-x^2=(x^2+x+1)(x^2-x+1),$$
$$\therefore \text{ the L.C.M.} = x(x+1)(x-1)(x^2+x+1)(x^2-x+1).$$

If the multiplications be performed the L.C.M. will be found to be x^7-x. It is customary, however, to leave the result in the factored form, as it is in this form that it is usually made use of.

EXERCISE 83

Find the H.C.F. and L.C.M. of:

1.* $4x^3y^2z,\ 8xy^4z^3,\ 12axy^2z$.

2. $x^2-y^2,\ xy-y^2,\ x^2-xy$.

3. $a^2-b^2,\ ab+b^2,\ a^2+2ab+b^2$.

4. $x^2-7x+12,\ x^2+2x-15,\ x^2-9$.

5. $a^2+8a+15,\ a^2-2a-35,\ a^2+3a-10$.

6. $3x^2-12x+12,\ 3x^2-12,\ 3x^2-3x-6$.

7. $x^2-xy+xz-yz,\ xy-y^2$.

8. $m^3-8,\ m^4n^2-4m^2n^2,\ 4m^2-16m+16$.

9. $6a^3-6b^3,\ 2a^3+2a^2b+2ab^2$.

10. $a^2+ab-ac,\ a^2+b^2-c^2+2ab$.

11. $a^2-b^2-c^2-2bc,\ b^2-c^2-a^2-2ca,\ c^2-a^2-b^2-2ab$.

12. $x^3+y^3,\ x^4+x^2y^2+y^4$.

13. $3x^2+7x-6,\ 3x^2-11x+6,\ 6x^2-13x+6$.

14. $10ax-2a+15cx-3c,\ 25x^2-1,\ 25x^2-10x+1$.

15. $x^3-5x^2+6x,\ x^3-3x^2+5x-15$.

16. $u^4-v^4,\ u^3-v^3,\ u^2-v^2,\ u-v$.

17. $x^3+2x^2-8x-16,\ x^3+3x^2-8x-24$.

18. Show that the product of $x^2-8x+15$ and x^2+x-12 is equal to the product of their H.C.F. and L.C.M.

19. The L.C.M. of a^2-5a+6 and $a^2\ \ldots\ -6$ is $a^3-3a^2-4a+12$. Supply the missing term.

20. Find two trinomials whose H.C.F. is $x-2y$ and whose L.C.M. is $x^3-7xy^2+6y^3$.

Ex. 1.—Find the H.C.F. and L.C.M. of

$$x^2+2x-3 \text{ and } x^3-8x+3.$$

Here x^2+2x-3 is readily factored, but none of the methods previously given will apply in factoring x^3-8x+3, except by using the factor theorem of art. 101.

The difficulty is, however, easily overcome thus:

$$x^2+2x-3=(x-1)(x+3).$$

If the expressions have a common factor it must evidently be either $x-1$ or $x+3$.

By using the factor theorem, find if $x-1$ or $x+3$ is a factor of x^3-8x+3.

When $x-1=0$ or $x=1$, $x^3-8x+3=1-8+3=-4$,

$$\therefore \quad x-1 \text{ is not a factor.}$$

When $x+3=0$ or $x=-3$, $x^3-8x+3=-27+24+3=0$,

$$\therefore \quad x+3 \text{ is a factor.}$$

How can we obtain the other factor of x^3-8x+3?

We now have $\quad x^2+2x-3=(x-1)(x+3)$,

and $\quad x^3-8x+3=(x+3)(x^2-3x+1)$.

$\therefore$ the H.C.F. $=x+3$,

and the L.C.M. $=(x+3)(x-1)(x^2-3x+1)$.

Ex. 2.—Find the H.C.F. and L.C.M. of

$$x^2-7x+10 \text{ and } x^3-6x^2+11x-6.$$

The factors of $x^2-7x+10$ are $(x-5)(x-2)$.

Here it is evident that $x-5$ is not a factor of the second expression, since its last term is -6, which is not divisible by 5.

Is $x-2$ a factor of $x^3-6x^2+11x-6$?

Complete the solution.

EXERCISE 89

Find the H.C.F. and L.C.M. of:

1.* x^2-3x+2, x^3-6x^2+8x-3.

2. a^2-6a+5, $a^3-19a^2+17a+1$.

3. x^3-2x^2+4x-8, $2x^3-7x^2+12$.

4. a^3-a^2+a-1, $3a^3-2a^2+5a-6$.

5. x^3+3x^2-4x, x^3-7x+6.

6. If $x-2$ is a common factor of

$$x^3+3x^2-9x-2 \text{ and } x^3-4x^2+3x+2,$$

find their L.C.M.

7. Reduce to lowest terms:

$$\frac{a^2-3ab+2b^2}{a^3-19ab^2+30b^3} \text{ and } \frac{x^3-2x^2-3x}{2x^4-14x^2-12x}.$$

8. Find two expressions of the third degree in x, whose H.C.F. is x^2-5x+6 and whose L.C.M. is $x^4-10x^3+35x^2-50x+24$.

133. Method of finding the H.C.F. of two expressions which can not be factored by the usual methods. From the preceding it is seen that the chief difficulty in finding the H.C.F. of two expressions is in factoring the given expressions.

If neither of the expressions can be factored by the usual methods, another method may be used which depends upon the same principle as that of finding the G.C.M. of two numbers in arithmetic.

134. Fundamental Theorem. This method of finding H.C.F. depends upon the following theorem:

If x is a common factor of any two quantities, then x is also a factor of the sum or difference of any multiples of those quantities.

Thus, x is a common factor of mx and nx.

Then $mx+nx$, $mx-nx$, $pmx+qnx$, $rmx-snx$, are each the sum or difference of multiples of mx and nx.

It is evident that each of these is divisible by x, the quotient in each case being found by division, thus:

$$x)\underline{mx+nx}\atop{\quad m\ \ +n},\qquad x)\underline{mx-nx}\atop{\quad m\ \ -n},\qquad x)\underline{pmx+qnx}\atop{\quad pm\ \ +qn},\qquad x)\underline{rmx-snx}\atop{\quad rm\ \ -sn}.$$

The way in which the theorem is applied is shown in the following examples.

Ex. 1.—Find the H.C.F. of

$$x^3+4x^2+4x+3 \text{ and } x^3+3x^2+4x+12.$$

Any common factor of these is a factor of their difference, which is x^2-9.

But

$$x^2-9=(x-3)(x+3),$$

$\therefore$ the H.C.F. is $x-3$ or $x+3$ or $(x-3)(x+3)$.

It is evident that $x-3$ is not a factor of either expression, since their terms are all positive. Therefore if they have a common factor it must be $x+3$.

By applying the factor theorem, or by division, we find that $x+3$ is a factor of each, and since it is the only common factor, it must be the H.C.F.

Ex. 2.—Find the H.C.F. of

$$3x^3-17x^2-5x+10 \text{ and } 3x^3-23x^2+23x-6.$$

Their difference $=6x^2-28x+16=2(x-4)(3x-2)$.

Now 2 is not a factor of either and may be discarded, also $x-4$ is not a factor, since 4 is not a factor of 10 nor of 6. Therefore if there is a common factor it must be $3x-2$.

Divide $3x-2$ into one of them and see if it divides evenly. If it does not there is no common factor but unity.

If it does divide evenly into one of them, it is not necessary to divide it into the other, for if it is a factor of one of them and also of their difference it must be a factor of the other.

Ex. 3.—Find the H.C.F. of

$$3x^3-13x^2+23x-21 \text{ and } 6x^3+x^2-44x+21.$$

Multiply the first by 2 and subtract the product from the second and we get

$$27x^2-90x+63=9(x-1)(3x-7).$$

Now since $9(x-1)(3x-7)$ is the difference of two multiples of the given expressions, it must contain all their common factors. Which of these factors may be discarded ? Complete the solution.

We might have obtained the H.C.F. thus :

The sum of the expressions is

$$9x^3-12x^2-21x=3x(x+1)(3x-7).$$

This expression contains all the common factors of the given expressions.

Complete the solution by this method.

The object in each case is to obtain from the given expressions *an expression of the second degree.* If this expression can not be factored, it must be the H.C.F., if there is any common factor other than unity. If it can be factored the H.C.F. can then be found either by the factor theorem or by ordinary division.

In obtaining the expression of the second degree, the last problem shows that it is sometimes easier to eliminate the *last terms* than the *first terms*.

Ex. 4.—Find the H.C.F. and L.C.M. of

$$6x^3-5x^2-8x+3 \text{ and } 4x^3-8x^2+x+3.$$

Eliminate the absolute terms and show that $2x-3$ is the H.C.F.

Since $2x-3$ is a factor of each, the other factors may be found by division, then

$$6x^3-5x^2-8x+3=(2x-3)(3x^2+2x-1),$$
$$4x^3-8x^2+x+3=(2x-3)(2x^2-x-1),$$
$$\therefore \text{ the L.C.M.}=(2x-3)(3x^2+2x-1)(2x^2-x-1).$$

Why is it unnecessary to factor $3x^2+2x-1$ and $2x^2-x-1$?

Ex. 5.—Find the H.C.F. of

$$x^4-4x^3+10x^2-11x+10, \qquad (1)$$

and

$$x^4-x^3-4x^2+19x-15. \qquad (2)$$

Subtract (1) from (2), and we get

$$3x^3-14x^2+30x-25. \qquad (3)$$

Multiply (1) by 3 and (2) by 2 and add to eliminate the absolute terms. Remove the factor x and we obtain

$$5x^3-14x^2+22x+5. \qquad (4)$$

The common factor we are seeking must be a factor of both (3) and (4).

Eliminate the absolute terms from (3) and (4) and show that the H.C.F. is x^2-3x+5.

Find also the L.C.M.

Ex. 6.—Find the H.C.F. of

$$8x^4+4x^3+4x^2-4x \text{ and } 6x^4+2x^3+2x^2-4x.$$

Here $4x$ is a factor of the first expression and $2x$ of the second, and therefore $2x$ is a common factor. Remove these simple factors and find the H.C.F. of the quotients, and show that the H.C.F. is $2x(x^2+x+1)$.

135. Product of the H.C.F. and L.C.M.

Suppose that x is the H.C.F. of mx and nx, so that m and n have no common factor.

Then the L.C.M. of mx and nx is mnx.

But $$x \times mnx = mx \times nx,$$

therefore *the product of any two quantities is equal to the product of their H.C.F. and L.C.M.*

Is a similar theorem true concerning any three quantities mx, nx and px?

If the H.C.F. of two quantities has been found, we might therefore find their L.C.M. by dividing their product by the H.C.F.

EXERCISE 90

Find the H.C.F. of:

1.* $x^3-7x^2+13x-15$, x^3-6x^2+x+20.

2. $a^3-10a^2+33a-36$, $a^3-2a^2-23a+60$.

3. $6x^3+10x^2+8x+4$, $6x^3-2x^2-4$.

4. $2x^3-5x^2-20x+9$, $2x^3+x^2-43x-9$.

5. $2b^3+5b^2-8b-15$, $4b^3-4b^2-9b+5$.

6. $3x^3+17x^2y-44xy^2-28y^3$, $6x^3-5x^2y-33xy^2+28y^3$.

7. $2a^3-3a^2-4a+4$, $3a^4-4a^3-10a+4$.

8. $2x^4-12x^3+19x^2-6x+9$, $4x^3-18x^2+19x-3$.

9. $18a^5b-3a^4b-12a^3b-3a^2b$, $12a^5c-6a^4c-9a^3c+3a^2c$.

10. x^3-x^2-2x+2, $x^4-3x^3+2x^2+x-1$.

Find the L.C.M. of:

11. x^3-7x-6, x^3-4x^2+4x-3.

12. $x^3+6x^2+11x+6$, $x^3+7x^2+14x+8$, $x^3+8x^2+19x+12$.

13. $2x^3+9x^2+7x-3$, $3x^3+5x^2-15x+4$.

14. $x^3-6x^2+11x-6$, $x^3-7x^2+14x-8$.

15. $20x^4+x^2-1$, $25x^4-10x^2+1$, $25x^4+5x^3-x-1$.

16. Find a value of x which will make $x^3-13x+12$ and x^3-6x^2-x+30 each equal to 0.

17. The L.C.M. of two numbers is 70 and the H.C.F. is 7. If one of the numbers is 14, find the other.

18. The H.C.F. of two expressions is $x-2$, the L.C.M. is $x^3-39x+70$. If one of the expressions is $x^2-7x+10$, find the other.

19. Two integers differ by 11. If they have a common factor, other than unity, what must it be ?

EXERCISE 91 (Review of Chapter XIV)

Find the H.C.F. and L.C.M. of :

1.* $x^2-20x+99$, $x^2-24x+143$, $x^2-21x+110$.

2. $x^2-15x+36$, x^3-27, x^3-3x^2-2x+6.

3. a^2-b^2, $a^2-2ab+b^2$, a^3-b^3.

4. x^3-2x^2-15x, $x^3+x^2-14x-24$.

5. $4a^3-12a^2-a+3$, $2a^3+a^2-18a-9$.

6. $x^2-ax\quad bx+ab$, $x^2-bx-cx+bc$.

7. $x^3-6\quad{}^2+11x-6$, x^3+4x^2+x-6.

8. $x^4+3x^3+3x^2+5x-12$, $x^4-4x^3-19x^2\quad x+12$.

9. $2a^4+15a^3+39a^2+40a+12$, $2a^4+9a^3-2a^2-39a-18$.

10. $x^4-6x^3y+13x^2y^2-12xy^3+4y^4$, $x^4+2x^3y-3x^2y^2-4xy^3+4y^4$.

11. $x^4+x^2y^2+y^4$, $x^4-2x^3y+3x^2y^2-2xy^3+y^4$.

12. Show that two consecutive integers can have no integral common factor except unity.

13. Two odd integers which differ by 6 have a common factor other than unity. What must it be ?

14. Find the H.C.F. of x^6+a^6 and $x^8+x^4a^4+a^8$.

15. If the H.C.F. of a and b is d, show that the L.C.M. is $\frac{ab}{d}$.

16. If a is the H.C.F. and b is the L.C.M. of three quantities, show that the product of the quantities is a^2b.

17. For what common values of x will

$$x^3-3x^2-x+3 \text{ and } x^4-4x^2+12x-9$$

both vanish ?

18. Find two expressions of the second degree in x, whose H.C.F. is $x-1$ and L.C.M. is $x^3-8x^2+17x-10$.

19 Reduce $\dfrac{18x^3-3x^2+2x+8}{12x^3+8x^2-7x+12}$ to lowest terms.

CHAPTER XV

FRACTIONS

In Chapter IX. fractions were introduced and simple examples of operations upon fractions.

In this Chapter the subject is extended and applications made to more complicated forms.

136. Changes in the Form of a Fraction. Both terms of a fraction may be multiplied or divided by the same quantity without altering the value of the fraction. As previously stated, the only exception to this rule is, that the quantity by which we multiply or divide must not be zero.

The rule might be stated in the symbolic form:

$$\frac{a}{b}=\frac{ma}{mb} \text{ or } \frac{na}{nb}=\frac{a}{b}.$$

The case in which the terms are multiplied or divided by -1 deserves special attention.

From the rule of signs for division $\frac{a}{-b}$ is seen to be the same as $-\frac{a}{b}$, so also is $\frac{-a}{b}$.

$$\therefore \quad \frac{-a}{b}=\frac{a}{-b}=-\frac{a}{b}.$$

Similarly, $$\frac{-a}{-b}=\frac{a}{b}=-\frac{-a}{b}=-\frac{a}{-b}.$$

It is thus seen, that *the value of a fraction is not changed by changing the signs of both of its terms; or by changing the sign of one of its terms and at the same time changing the sign before the fraction.*

Since $(a-b)\times(-1)=-a+b$ or $b-a$, it is seen that $a-b$ and $b-a$ differ only in sign, or that each one is equal to the other multiplied by -1.

That is, $a-b=-(b-a)$ and $b-a=-(a-b)$.

Then $\quad \dfrac{a-b}{c-d}=\dfrac{(a-b)\times(-1)}{(c-d)\times(-1)}=\dfrac{b-a}{d-c}=-\dfrac{a-b}{d-c}=-\dfrac{b-a}{c-d}.$

Also, since $\quad (-a)\times(-b)=(+a)\times(+b)=ab,$

it follows that $\quad (a-b)(c-d)=(b-a)(d-c),$

$$\frac{m}{(a-b)(c-d)}=\frac{m}{(b-a)(d-c)}=\frac{-m}{(a-b)(d-c)}=-\frac{m}{(b-a)(c-d)},$$

$$\frac{(a-x)(b-y)}{(b-x)(a-y)}=\frac{(x-a)(y-b)}{(x-b)(y-a)}=\frac{(a-x)(y-b)}{(b-x)(y-a)}=\text{etc.}$$

EXERCISE 92 (1-29, Oral)

Express these fractions in their simplest forms with no negative signs in either term:

1. $\dfrac{-2}{4}$. 2. $\dfrac{4}{-2}$. 3. $\dfrac{-6}{-9}$. 4. $\dfrac{-3a}{-a}$.

5. $\dfrac{-a}{y}$. 6. $\dfrac{5}{-m}$. 7. $-\dfrac{-4ab}{2b}$. 8. $\dfrac{-ax}{-bx}$.

9. $\dfrac{-3\times-5}{7}$. 10. $\dfrac{-a\times b}{-b}$. 11. $-\dfrac{-a\,.\,-b}{-c}$. 12. $\dfrac{-x\,.\,-y}{-a\,.\,-b}$.

Express with the numerator a

13. $\dfrac{b-a}{-3}$. 14. $-\dfrac{b-a}{-b}$. 15. $\dfrac{b-a}{x-y}$. 16. $\dfrac{b-a}{-3(c-d)}$.

Express with the denominator $c-d$:

17. $\dfrac{-5}{d-c}$. 18. $\dfrac{-x\,.\,-y}{d-c}$. 19. $\dfrac{a-b}{d-c}$ 20. $\dfrac{-m(x-y)}{d-}$.

Express with the positive sign before the fraction:

21. $-\dfrac{-4}{7}$. 22. $-\dfrac{4}{-7}$. 23. $-\dfrac{a}{b}$. 24. $-\dfrac{a-b}{c}$.

25. $-\dfrac{x}{a-b}$. 26. $-\dfrac{a+b}{a-b}$. 27. $-\dfrac{x-2}{x-y}$. 28. $-\dfrac{c-d}{c+d}$

29. What is the relation between

$\dfrac{a}{x-y}$ and $\dfrac{a}{y-x}$, $\quad \dfrac{a+b}{a-b}$ and $\dfrac{b+a}{b-a}$, $\quad \dfrac{b-a}{-3}$ and $\dfrac{a-b}{3}$?

30. Write $\dfrac{(p-q)(q-r)}{(x-y)(y-z)}$ in four equivalent forms, with the positive sign before the fraction.

31. Which of the following are equal in value:

$$(a-b)(b-c)(c-a),\ (b-a)(c-b)(a-c),\ (a-b)(c-b)(a-c),$$
$$(a-b)(b-c)(a-c),\ (b-a)(b-c)(a-c)\ ?$$

137. Reduction of Fractions to Lowest Terms. The formula $\dfrac{ax}{bx}=\dfrac{a}{b}$ may be used to reduce a fraction to its lowest terms, by dividing both terms by all the common factors.

Ex. 1.—Reduce $\dfrac{x^3+y^3}{x^4+x^2y^2+y^4}$.

$$x^3+y^3=(x+y)(x^2-xy+y^2).$$
$$x^4+x^2y^2+y^4=(x^2+y^2)^2-x^2y^2=(x^2+xy+y^2)(x^2-xy+y^2),$$
$$\therefore \text{ the fraction }=\frac{x+y}{x^2+xy+y^2}.$$

Ex. 2.—Reduce $\dfrac{a^2+b^2-c^2+2ab}{a^2-b^2+c^2+2ac}$.

$$a^2+b^2-c^2+2ab=(a+b)^2-c^2=(a+b+c)(a+b-c).$$

Complete the reduction.

Ex. 3.—Reduce $\dfrac{x^2-11x+28}{2x^3-6x^2+7x-60}$.

$$x^2-11x+28=(x-4)(x-7).$$

Which of these factors can not divide into the denominator? Complete the reduction.

Ex. 4.—Reduce $\dfrac{3x^3-15x^2-19x+6}{6x^3+3x^2-5x+1}$.

Here the factors of neither term can be readily obtained, so the common factor must be found by the method of art. 134.

Eliminate the x^3 and we obtain

$$33x^2+33x-11 \text{ or } 11(3x^2+3x-1).$$

This expression must contain any common factor of both terms. Since $3x^2+3x-1$ can not be factored, what conclusion can be drawn? Complete the reduction.

EXERCISE 99

Reduce to lowest terms:

1. $\dfrac{a^2+3a+2}{a^2+5a+6}$.

2. $\dfrac{x^2+7xy-8y^2}{x^2+5xy-24y^2}$.

3. $\dfrac{6x^2+x-1}{4x^2+8x+3}$.

4. $\dfrac{x^2+2xy+y^2-z^2}{x^2-y^2-2yz-z^2}$.

5. $\dfrac{a^3+a^2b+ab^2+b^3}{(a-b)^2+4ab}$.

6. $\dfrac{2-3y-2y^2}{4-5y-6y^2}$.

7.* $\dfrac{a^2+2a+1}{a^3+2a^2+2a+1}$.

8. $\dfrac{x^3-x^2-2x}{x^3-3x^2+4}$.

9. $\dfrac{a^2-4a+3}{4a^3-9a^2-15a+18}$.

10. $\dfrac{3x^2-3x-18}{6x^5-12x^4-18x^3}$.

11. $\dfrac{2x^3-x^2+2x-3}{2x^3+3x^2+4x+3}$.

12. $\dfrac{3x^3+4x^2-6x-8}{36x^3+27x^2-40x-16}$.

13. $\dfrac{a^3+a^2-3a-3}{a^5-a^4-2a^3+2a^2-3a-5}$.

14. $\dfrac{2x^4-4x^3-2x^2-12x}{4x^4+2x^3+6x^2-4x}$.

138. Addition and Subtraction. In adding or subtracting fractions we should be careful to note whether any of the given fractions can be reduced to lower terms. When the result is obtained we should examine it to see if it can be reduced.

Ex. 1.—Simplify $\dfrac{x-y}{x}+\dfrac{2y}{x-y}-\dfrac{xy^2+y^3}{x^3-xy^2}$.

The expression

$$
\begin{aligned}
&=\frac{x-y}{x}+\frac{2y}{x-y}-\frac{y^2(x+y)}{x(x-y)(x+y)},\\
&=\frac{x-y}{x}+\frac{2y}{x-y}-\frac{y^2}{x(x-y)}.\\
&=\frac{(x-y)^2+2xy-y^2}{x(x-y)},\\
&=\frac{x^2}{x(x-y)}=\frac{x}{x-y}.
\end{aligned}
$$

The form of the last fraction in the given expression should prompt the pupil to examine whether it can be reduced.

Ex. 2.—Simplify $\frac{1}{x-2}+\frac{1}{x^2-3x+2}-\frac{2}{x^2-4x+3}$.

The expression

$$=\frac{1}{x-2}+\frac{1}{(x-1)(x-2)}-\frac{2}{(x-3)(x-1)},$$

$$=\frac{(x-1)(x-3)+x-3-2(x-2)}{(x-2)(x-1)(x-3)},$$

$$=\frac{x^2-5x+4}{(x-2)(x-1)(x-3)}=\frac{(x-4)(x-1)}{(x-2)(x-1)(x-3)}=\frac{x-4}{(x-2)(x-3)}.$$

EXERCISE 94

Simplify:

1.* $\frac{a}{a+b}+\frac{b}{a-b}$.

2. $\frac{1}{x-y}-\frac{1}{x+y}$.

3. $\frac{x+y}{x-y}-\frac{x-y}{x+y}$.

4. $\frac{a-4}{a-2}-\frac{a-7}{a-5}$.

5. $\frac{2a^2}{a^2-b^2}-\frac{2a}{a+b}$.

6. $\frac{a^2}{a-a^3}-\frac{a}{1+a^2}$.

7. $\frac{2x^2}{x^2-y^2}-\frac{2x^2}{x^2+xy}$.

8. $\frac{x-y}{x^2-y^2}+\frac{1}{2x-y}$.

9. $\frac{x^2-4y^2}{x^2+2xy}-\frac{x-2y}{x}$.

10. $\frac{1}{x^2+9x+20}-\frac{1}{x^2+12x+35}$.

11. $\frac{2x}{x+y}-\frac{3y}{x-y}+\frac{x^2+y^2}{x^2-y^2}$.

12. $\frac{x^2-3x-10}{x^2-8x+15}-\frac{x^2+2x-3}{x^2-3x+2}$.

13. $\frac{x-y}{y}+\frac{2x}{x-y}-\frac{x^3+x^2y}{x^2y-y^3}$.

14. $\frac{a^2-ab+b^2}{a-b}-\frac{a^2+ab+b^2}{a+b}$.

15. $\frac{x+y}{x-y}+\frac{4xy}{x^2-y^2}-\frac{y-x}{x+y}$.

16. $\frac{a-b}{2(a+b)}+\frac{a+b}{2(a-b)}-\frac{a^2+b^2}{a^2-b^2}$.

17. $\frac{3x^2-8}{x^3-1}-\frac{5x+7}{x^2+x+1}+\frac{2}{x-1}$.

18. $\frac{1}{2a-3b}+\frac{2a+9b}{4a^2-9b^2}+\frac{1}{2a+3b}$.

19. $\frac{1}{x-1}-\frac{2}{x-2}+\frac{1}{x-3}$.

20. $\frac{1}{x^2-3x+2}+\frac{3}{x^2-7x+10}-\frac{4}{x^2-6x+5}$.

21. $\frac{(a+b)(b+c)(c+a)}{abc}-\frac{a+b}{c}-\frac{b+c}{a}-\frac{c+a}{b}$

22. $\dfrac{a^2-b^2+2bc-c^2}{b^2-c^2+2ac-a^2}-\dfrac{c^2+2ca+a^2-b^2}{b^2+2bc+c^2-a^2}$

23. $\dfrac{a^2-(b-c)^2}{(a+c)^2-b^2}+\dfrac{b^2-(c-a)^2}{(b+a)^2-c^2}+\dfrac{c^2-(a-b)^2}{(b+c)^2-a^2}$

24. $\dfrac{a^2-2a}{a^2-a-2}-\dfrac{3a}{6a-4}+\dfrac{5a}{6a^2+2a-4}.$

25. $\dfrac{1}{x-y}-\dfrac{1}{x+y}$ and $\dfrac{1}{x-y}-\dfrac{1}{x+y}-\dfrac{2y}{x^2+y^2}$

26. $\dfrac{1}{x-y}-\dfrac{1}{x+y}-\dfrac{2y}{x^2+y^2}-\dfrac{4y^3}{x^4+y^4}.$

27. $\dfrac{1}{3-x}-\dfrac{1}{3+x}-\dfrac{2x}{9+x^2}.$

28. $\dfrac{a}{a-b}+\dfrac{a}{a+b}+\dfrac{2a^2}{a^2+b^2}+\dfrac{4a^2b^2}{a^4-b^4}.$

29. $\dfrac{1}{4-4x}-\dfrac{1}{4+4x}+\dfrac{x}{2+2x^2}-\dfrac{x}{1+x^4}.$

30. Solve $\dfrac{x}{x^2+5x+6}+\dfrac{15}{x^2+9x+14}-\dfrac{12}{x^2+10x+21}=\dfrac{1}{4}.$ (Verify.)

139. Special Types in Addition and Subtraction.
We have already seen that

$$b-a=-(a-b) \text{ and } a-b=-(b-a),$$
$$\text{or } (a-b)\div(b-a)=-1.$$

When $a-b$ and $b-a$ occur in the factored denominators of different fractions, which are to be combined, it is not necessary to include both of them in the L.C.D.

Ex. 1.—Simplify $\dfrac{a}{a-b}+\dfrac{b}{b-a}.$

Here only one factor is required in the L.C.D. and we may use either $a-b$ or $b-a$.

f we decide to use $a-b$, then it is better to change the second fraction into the form $-\dfrac{b}{a-b}.$

Then $$\frac{a}{a-b}+\frac{b}{b-a}=\frac{a}{a-b}-\frac{b}{a-b}=\frac{a-b}{a-b}=1.$$

Ex. 2.—Simplify $\frac{4}{x-1}-\frac{3}{x+1}+\frac{x-3}{1-x^2}$.

The denominator of the last fraction should be changed to x^2-1, so as to be the product of $x-1$ and $x+1$.

$$\text{The expression} = \frac{4}{x-1}-\frac{3}{x+1}-\frac{x-3}{x^2-1},$$
$$= \frac{4(x+1)-3(x-1)-(x-3)}{(x-1)(x+1)} = \frac{10}{(x-1)(x+1)}.$$

Ex. 3.—Simplify

$$\frac{1}{(x-1)(x-2)}+\frac{2}{(x-2)(3-x)}-\frac{3}{(x-3)(1-x)}.$$

Here there are only three factors $x-1$, $x-2$, $x-3$, required in the L.C.D.

We therefore change the second and third fractions so that the given expression

$$= \frac{1}{(x-1)(x-2)}-\frac{2}{(x-2)(x-3)}+\frac{3}{(x-3)(x-1)}.$$

Complete the simplification.

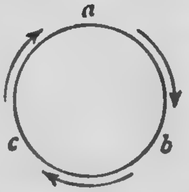
140. Cyclic Order. Suppose we wish to simplify

$$\frac{b+c}{(a-b)(a-c)}+\frac{c+a}{(b-c)(b-a)}+\frac{a+b}{(c-a)(c-b)}.$$

The L.C.D. in this case will contain three factors and it might be written in different forms as

$$(a-b)(b-c)(c-a),\ (a-b)(a-c)(b-c),\ \text{etc.}$$

The pupil is advised to write the factors in what is called **cyclic order**.



If we arrange the letters on the circumference of a circle, as in the diagram, and follow the direction of the arrows we see that a is followed by b, b by c, and c by a.

Thus, if we write $a-b$ as the first factor, then changing a to b, b to c, and c to a, we write the second factor $b-c$ and the third $c-a$.

If we write the L.C.D. as $(a-b)(b-c)(c-a)$, we should change the fractions so that these factors appear in the denominators.

The given expression then

$$= -\frac{b+c}{(a-b)(c-a)} - \frac{c+a}{(b-c)(a-b)} - \frac{a+b}{(c-a)(b-c)},$$

$$= \frac{-(b+c)(b-c)-(c+a)(c-a)-(a+b)(a-b)}{(a-b)(b-c)(c-a)}$$

$$= \frac{-b^2+c^2-c^2+a^2-a^2+b^2}{(a-b)(b-c)(c-a)} = 0.$$

Ex.—Simplify $\dfrac{bc}{(a-b)(a-c)} + \dfrac{ca}{(b-c)(b-a)} + \dfrac{ab}{(c-a)(c-b)}$

Proceed as in the preceding example and you should get the result

$$\frac{-b^2c+bc^2-c^2a+ca^2-a^2b+ab^2}{(a-b)(b-c)(c-a)}.$$

This fraction is equal to unity, for the numerator is equal to the denominator. Prove that this is true.

EXERCISE 95

Simplify:

1.* $\dfrac{1}{a^2+ax} - \dfrac{1}{x^2+ax}$.

2. $\dfrac{6}{2a-3b} - \dfrac{3}{3b-2a}$.

3. $\dfrac{x+3}{x-2} - \dfrac{x-3}{2-x}$.

4. $\dfrac{x}{x^2-9y^2} + \dfrac{1}{3y-x}$

5. $\dfrac{2}{a^2-ab} - \dfrac{3}{b^2-ab}$.

6. $\dfrac{x+a}{x-a} - \dfrac{x^2-a^2}{a^2-ax}$.

7. $\dfrac{2}{x-1} - \dfrac{3}{x+1} - \dfrac{x^2-3}{1-x^2}$.

8. $\dfrac{5}{x-2} - \dfrac{4}{2+x} + \dfrac{16}{4-x^2}$.

9. $\dfrac{x}{x-y} - \dfrac{y}{x+y} + \dfrac{y^2}{y^2-x^2}$

10. $\dfrac{3}{x+4} - \dfrac{3}{x-4} - \dfrac{24}{16-x^2}$.

11. $\dfrac{1}{b+3a} - \dfrac{1}{3a-b} + \dfrac{6a}{b^2-9a^2}$.

12. $\dfrac{3a+2x}{3a-2x} - \dfrac{3a-2x}{3a+2x} + \dfrac{16x^2}{4x^2-9a^2}$.

13. $\dfrac{1}{x-1} - \dfrac{4}{1-x} - \dfrac{8}{1+x} + \dfrac{3x+7}{x^2-1}$.

14. $\frac{a}{c(a-b)}-\frac{c}{a(b-a)}$.

15. $\frac{1}{(a-b)(c-a)}+\frac{1}{(b-a)(c-b)}$.

16. $\frac{a}{(x-a)(a-b)}+\frac{b}{(x-b)(b-a)}$.

17. $\frac{2}{y^2-x^2}+\frac{2}{x^2-2xy+y^2}+\frac{1}{x^2+xy}$.

18. $\frac{3-x}{1-3x}-\frac{3+x}{1+3x}-\frac{1-16x}{9x^2-1}$.

19. $\frac{2}{x^2-8x+15}+\frac{2}{x^2-4x+3}+\frac{4}{6x-x^2-5}$.

20. $\frac{a+b}{(b-c)(c-a)}+\frac{b+c}{(c-a)(a-b)}+\frac{c+a}{(a-b)(b-c)}$.

21. $\frac{1}{(a-b)(a-c)}+\frac{1}{(b-c)(b-a)}+\frac{1}{(c-a)(c-b)}$.

22. $\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-c)(b-a)}+\frac{c^2}{(c-a)(c-b)}$.

23. $\frac{z}{(x-y)(x-z)}+\frac{x}{(y-z)(y-x)}+\frac{y}{(z-x)(z-y)}$.

24. $\frac{ax-bc}{(a-b)(a-c)}+\frac{bx-ca}{(b-c)(b-a)}+\frac{cx-ab}{(c-a)(c-b)}$.

25. $\frac{a^2}{(a^2-b^2)(a^2-c^2)}+\frac{b^2}{(b^2-c^2)(b^2-a^2)}+\frac{c^2}{(c^2-a^2)(c^2-b^2)}$.

26. $\frac{bc(a+d)}{(a-b)(a-c)}+\frac{ca(b+d)}{(b-c)(b-a)}+\frac{ab(c+d)}{(c-a)(c-b)}$.

27. $\frac{1}{a-b}-\frac{1}{2(b+a)}-\frac{a+3b}{2(a^2+b^2)}+\frac{4b^3}{b^4-a^4}$.

28. $\left(\frac{1}{x-5}+\frac{1}{x+5}\right)-\left(\frac{1}{x+3}+\frac{1}{x-3}\right)$.

29. $\frac{1}{a+4}-\frac{3}{a+3}+\frac{3}{a+2}-\frac{1}{a+1}$. (Check when $a=1$.)

30. $\frac{1}{x-5}+\frac{1}{x+3}-\frac{1}{x-1}-\frac{1}{x+1}$. (Check when $x=2$.)

141. Multiplication and Division. The ordinary cases in multiplication and division of fractions have been treated in art. 74. Some special forms which appear are illustrated in the following examples.

Ex. 1.—Multiply $a+\dfrac{ax}{a-x}$ by $a-\dfrac{ax}{a+x}$.

Here the mixed expressions should be reduced to the fractional form before multiplying.

The product $=\dfrac{a^2-ax+ax}{a-x}\times\dfrac{a^2+ax-ax}{a+x}=\dfrac{a^4}{a^2-x^2}$.

Ex. 2.—Multiply $\dfrac{a}{b}+\dfrac{b}{a}+1$ by $\dfrac{a}{b}+\dfrac{b}{a}-1$.

Multiply this in the ordinary way, by multiplying each term of the one by each term of the other.

We should recognize that the first expression is the sum of $\dfrac{a}{b}+\dfrac{b}{a}$ and 1 and the second is the difference.

The product $=\left(\dfrac{a}{b}+\dfrac{b}{a}\right)^2-1=\dfrac{a^2}{b^2}+2+\dfrac{b^2}{a^2}-1=\dfrac{a^2}{b^2}+1+\dfrac{b^2}{a^2}$.

Or, we might proceed as in Ex. 1, thus:

$$\frac{a^2+b^2+ab}{ab}\times\frac{a^2+b^2-ab}{ab}=\frac{(a^2+b^2)^2-a^2b^2}{a^2b^2}=\frac{a^4+a^2b^2+b^4}{a^2b^2}.$$

This result is seen to be the same as $\dfrac{a^2}{b^2}+1+\dfrac{b^2}{a^2}$, and the answer may be given in either form.

Ex. 3.—Divide $\dfrac{x^2}{y^3}+\dfrac{1}{x}$ by $\dfrac{x}{y^2}-\dfrac{1}{y}+\dfrac{1}{x}$.

The dividend $=\dfrac{x^3+y^3}{xy^3}$, the divisor $=\dfrac{x^2-xy+y^2}{xy^2}$,

$\therefore$ the quotient $=\dfrac{x^3+y^3}{xy^3}\times\dfrac{xy^2}{x^2-xy+y^2}=\dfrac{x+y}{y}$.

Divide in the ordinary way and get the quotient $\dfrac{x}{y}+1$.

We must not make the error of thinking that we can invert the divisor, or take the reciprocal of it, by inverting each term of it, and change the problem to one in multiplication, thus:

$$\left(\frac{x^2}{y^2}+\frac{1}{x}\right)\times\left(\frac{y^2}{x}-y+x\right).$$

The reciprocal of $\frac{1}{a+b}$ is $a+b$, but the reciprocal of $\frac{1}{a}+\frac{1}{b}$ is not $a+b$.

For $\frac{1}{a}+\frac{1}{b}=\frac{a+b}{ab}$ and its reciprocal is $\frac{ab}{a+b}$.

EXERCISE 96

Multiply:

1.* $\frac{x^2-x-6}{x^2+4x+4}$, $\frac{x^2-2x-8}{x^2-7x+12}$.

2. $\frac{a^2+b^2}{a^2-ab}$, $\frac{ab-b^2}{a^4-b^4}$, $\frac{a}{b}$.

3. $\frac{x^2+2x-15}{x^2+8x-33}$, $\frac{x^2+7x-44}{x^2+9x+20}$.

4. $\frac{a^4-b^4}{a^2-2ab+b^2}$, $\frac{a-b}{a^2+ab}$.

5. $1-\frac{a-b}{a+b}$, $1+\frac{a+b}{a-b}$.

6. $y+\frac{xy}{y-x}$, $y-\frac{xy}{x+y}$, $\frac{y^2-x^2}{y^2+x^2}$.

7. $x^2+1+\frac{1}{x^2}$, $x^2-1+\frac{1}{x^2}$.

8. $a^2+2+\frac{2}{a^2}$, $a^2-2+\frac{2}{a^2}$.

9. $b+\frac{bx}{a}$, $1-\frac{a}{a+x}$, $\frac{a}{bx}$.

10. $\frac{a^3-b^3}{a^3+b^3}$, $\frac{a+b}{a-b}$, $\frac{a^2-ab+b^2}{a^2+ab+b^2}$.

Divide:

11. $\frac{x^2-11x+30}{x^2-6x+9}$ by $\frac{x^2-5x}{x^2-3x}$.

12. $\frac{a^2-b^2}{x^2-xy}$ by $\frac{a^2+ab}{xy-y^2}$.

13. $\frac{a^2+b^2-c^2+2ab}{c^2-a^2-b^2+2ab}$ by $\frac{a+b+c}{b+c-a}$.

14. $\frac{x^2}{y^2}-\frac{y^2}{x^2}+\frac{x}{y}-\frac{y}{x}$ by $\frac{x}{y}-\frac{y}{x}$.

15. $\frac{x^3}{y^3}+\frac{y^3}{x^3}$ by $\frac{x}{y}+\frac{y}{x}$.

16. $\frac{x^6-y^6}{x^4-y^4}$ by $\frac{x^3-y^3}{x-y}$.

Simplify:

17. $\frac{a+1}{a^2-2a}\times\frac{a+2}{a^2-a}\div\frac{a^2-4}{a^4-a^3}$.

18. $\frac{a^2-4}{a^2+5a}\div\frac{a^2+2a}{a^2-25}$.

19. $\dfrac{2x^2+x-1}{x^2-4x+3}\times\dfrac{2x^2-5x+3}{6x^2+x-2}\div\dfrac{2x^2-7x+6}{3x^2-7x-6}$.

20. $\left(\dfrac{a}{a-x}-\dfrac{x}{a+x}+\dfrac{2ax}{a^2-x^2}\right)\div(a+x)^2$.

21. $\left(1+\dfrac{b}{a}\right)\left(1+\dfrac{c}{a}\right)\div\left(1+\dfrac{a}{b}\right)\left(1+\dfrac{a}{c}\right)$.

22. $\dfrac{(a+b)^2-(c+d)^2}{(a+c)^2-(b+d)^2}\div\dfrac{(a-c)^2-(d-b)^2}{(a-b)^2-(d-c)^2}$.

23. $\dfrac{a^3-64}{a^2+24a+128}\times\dfrac{a^2+12a-64}{a^2-64}\div\dfrac{a^2-16a+64}{a^2+4a+16}$.

24. $\left(\dfrac{5a}{a-6b}-\dfrac{2b}{3a-2b}\right)\div\left(\dfrac{2a}{a+2b}-\dfrac{2b-a}{2b-3a}\right)$.

142. Complex Fractions. A complex fraction is one which contains fractional forms in either the numerator or denominator or both.

Thus, $\dfrac{\frac{a}{b}}{\frac{c}{d}}$ is a complex fraction and is, of course, only another way of writing $\dfrac{a}{b}\div\dfrac{c}{d}$. It is simplified in the usual way by changing it into $\dfrac{a}{b}\times\dfrac{d}{c}$ which equals $\dfrac{ad}{bc}$.

A complex fraction may sometimes be easily simplified by multiplying both terms by the same quantity.

Thus, $\dfrac{2-\frac{3}{4}a}{b}=\dfrac{8-3a}{4b}$ on multiplying each term by 4.

$$\frac{\dfrac{a+2b}{a+b}+\dfrac{a}{b}}{\dfrac{a+2b}{b}-\dfrac{a}{a+b}}=\frac{b(a+2b)+a(a+b)}{(a+2b)(a+b)-ab}=\frac{a^2+2ab+2b^2}{a^2+2ab+2b^2}=1.$$

Here both terms were multiplied by $b(a+b)$.

If the L.C.D. is not the same for both terms of the fraction, it is usually better to simplify the terms separately.

Ex.—Simplify $\dfrac{\dfrac{x+1}{x+2}-\dfrac{x-1}{x-2}}{\dfrac{x-2}{x+2}-\dfrac{x+3}{x-3}}$.

$$\text{The numerator} = \frac{(x+1)(x-2)-(x-1)(x+2)}{(x+2)(x-2)} = \frac{-2x}{(x+2)(x-2)}.$$

$$\text{The denominator} = \frac{(x-2)(x-3)-(x+2)(x+3)}{(x+2)(x-3)} = \frac{-10x}{(x+2)(x-3)}.$$

$$\therefore \text{ the fraction} = \frac{-2x}{(x+2)(x-2)} \times \frac{(x+2)(x-3)}{-10x} = \frac{x-3}{5(x-2)}.$$

EXERCISE 97

Simplify :

1.* $\dfrac{\dfrac{6a}{5b}}{12c}$.

2. $\dfrac{6a}{\dfrac{5b}{12c}}$.

3. $\dfrac{\dfrac{1}{a-b}}{a+b}$.

4. $\dfrac{\dfrac{1}{x}-\dfrac{1}{y}}{x-y}$.

5. $\dfrac{\dfrac{1}{1+x}}{1-\dfrac{1}{1+x}}$.

6. $\dfrac{\dfrac{1}{a+b}+\dfrac{1}{a-b}}{\dfrac{1}{a+b}-\dfrac{1}{a-b}}$.

7. $\dfrac{x+y-\dfrac{x^2+y^2}{x+y}}{x+y-\dfrac{2xy}{x+y}}$.

8. $\dfrac{a-2-\dfrac{a^2-5a}{a-3}}{a+\dfrac{3a}{a-3}}$.

9. $\dfrac{(x+3y)^2-(x-3y)^2}{(3x+y)^2-(3x-y)^2}$.

10. $\dfrac{2+xy+\dfrac{1}{xy}}{x^2y^2-1}$.

11. $\dfrac{a+b-\dfrac{a^3}{a^2-ab+b^2}}{a+b-\dfrac{b^3}{a^2-ab+b^2}}$.

12. $\dfrac{\dfrac{c}{a+b}-\dfrac{a}{b+c}}{\dfrac{a}{b+c}-\dfrac{b}{c+a}}$.

13. $\dfrac{3}{2x+3-\dfrac{3}{1-\dfrac{x}{x+6}}}$.

14. $\dfrac{a^2}{\dfrac{a^2+b^2}{a+b}-b}+\dfrac{b^2}{\dfrac{a^2+b^2}{a+b}-a}$.

15. $\dfrac{\dfrac{a}{b^2}+\dfrac{b}{a^2}}{\dfrac{1}{a^2}-\dfrac{1}{ab}+\dfrac{1}{b^2}}$.

16. Find the value of $\dfrac{x-y}{x+y}$ when $x=\dfrac{a+b}{a-b}$, $y=\dfrac{a-b}{a+b}$.

17. Find the value of $\dfrac{1-2a+b}{1+2a+b}$ when $a=\dfrac{x-y}{x+y}$, $b=\dfrac{(x-y)^2}{(x+y)^2}$.

EXERCISE 98 (Review of Chapter XV)

Simplify :

1.* $\dfrac{a}{a+b}-\dfrac{a}{b-a}+\dfrac{2a^2}{a^2+b^2}$.

2. $\dfrac{\dfrac{a}{b}}{c}-\dfrac{a}{\dfrac{b}{c}}$.

3. $\dfrac{x}{x+3y}-\dfrac{y}{3y-x}-\dfrac{(x-y)^2}{x^2-9y^2}$.

4. $\dfrac{2}{\dfrac{1}{x}-\dfrac{1}{y}}+\dfrac{y}{1-\dfrac{y}{x}}-\dfrac{x}{1-\dfrac{x}{y}}$.

5. $\dfrac{1}{1-x}+\dfrac{1}{1+x}+\dfrac{2}{1+x^2}+\dfrac{4}{1+x^4}$.

6. $\dfrac{x}{x-1}-1-\dfrac{1}{x^2-x}+\dfrac{1}{x}$.

7. $\left\{1+\dfrac{(a-b)^2}{4ab}\right\}\div\left(1+\dfrac{b^2+a^2}{2ab}\right)$.

8. $\dfrac{(1-x)^2+(1+x)^2}{(1+x)^2-(1-x)^2}$.

9. $\left(1-\dfrac{2bc}{b^2+c^2}\right)\left(b-c+\dfrac{2bc}{b-c}\right)\left(1-\dfrac{bc-c^2}{b^2}\right)\dfrac{b^2}{b^3+c^3}$.

10. $\dfrac{1}{x+1}-\dfrac{1}{x^2+3x+2}+\dfrac{1}{x^3+6x^2+11x+6}$.

11. $\dfrac{a^2-ab}{a^3-b^3}+\dfrac{ab+b^2}{a^3+b^3}+\dfrac{2a^2b}{a^4+a^2b^2+b^4}$.

12. $\dfrac{a^6-b^6}{a^2+2ab+b^2}\times\dfrac{a^2+b^2}{a^2-b^2}\times\dfrac{a+b}{a^4+a^2b^2+b^4}$.

13. $\dfrac{a+x}{a^2+ax+x^2}+\dfrac{a-x}{a^2-ax+x^2}+\dfrac{2x^3}{a^4+a^2x^2+x^4}$.

14. $(1-a^3)\div\left\{(1-a)^2-(a^3-1)\div\left(a+\dfrac{1}{a+1}\right)\right\}$.

15. $\dfrac{b+c-a}{(a-b)(a-c)}+\dfrac{c+a-b}{(b-c)(b-a)}+\dfrac{a+b-c}{(c-a)(c-b)}$.

16. $\dfrac{x^3}{(x-y)(x-z)}+\dfrac{y^3}{(y-z)(y-x)}+\dfrac{z^3}{(z-x)(z-y)}$.

17. $\dfrac{4x+6}{6x^2+5x-6}+\dfrac{3x-9}{2x^2-3x-9}-\dfrac{15x-10}{9x^2-12x+4}$.

18. Express the product of

$$\frac{8-6x+x^2}{1+x} \text{ and } \frac{1}{(x-1)(2-x)}-\frac{1}{(x-2)(4-x)}-\frac{1}{(x-4)(1-x)}$$

as a fraction in its lowest terms.

19. How can you show mentally that 3 is the sum of

$$\frac{x}{x+a}+\frac{x}{x+b}+\frac{x}{x+c} \text{ and } \frac{a}{x+a}+\frac{b}{x+b}+\frac{c}{x+c}?$$

20. Divide $\dfrac{2x}{x+y}+\dfrac{y}{x-y}-\dfrac{y^2}{x^2-y^2}$ by $\dfrac{1}{x+y}+\dfrac{x}{x^2-y^2}$.

21. Divide $\dfrac{a^2-(b+c)^2}{a^2-(b-c)^2}\times\dfrac{b^2-(c-a)^2}{b^2-(c+a)^2}$ by $\dfrac{(a-b-c)^2}{(a-b+c)^2}$.

22. Divide $\dfrac{a-1}{a}+\dfrac{b-1}{b}+\dfrac{c-1}{c}-1$ by $2-\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)$.

23. Show that $\dfrac{(x+y)(1-xy)}{(1-xy)^2-(x+y)^2}=\dfrac{x(1-y^2)+y(1-x^2)}{(1-x^2)(1-y^2)-4xy}$.

24. If $a=\dfrac{2}{2-b}$, $b=\dfrac{2}{2-c}$, $c=\dfrac{2}{2-d}$, $d=\dfrac{2}{2-x}$, prove $a=x$.

25. Find the product of

$$\frac{1+x^2}{1+x^3} \text{ and } 1+x+\frac{1}{x-1+\dfrac{x}{x-1}}.$$

26. Subtract $\dfrac{a+10}{b+10}$ from $\dfrac{a}{b}$ and determine which fraction is the greater if a is greater than b and if both a and b are positive.

27. Add $\dfrac{(1+ab)(1+ac)}{(a-b)(a-c)}$, $\dfrac{(1+bc)(1+ba)}{(b-c)(b-a)}$, $\dfrac{(1+ca)(1+cb)}{(c-a)(c-b)}$.

28. Show that $\dfrac{1}{x-a}+\dfrac{1}{x-b}$ has the same value when $x=a+b$ as it has when $x=\dfrac{2ab}{a+b}$.

29. Prove that the product of any two quantities is equal to their sum divided by the sum of their reciprocals.

30. If $x=\frac{b-c}{a}$, $y=\frac{c-a}{b}$, $z=\frac{a-b}{c}$, prove that

$$xyz+x+y+z=0.$$

31. If $a=\frac{x-y}{x+y}$, $b=\frac{y-z}{y+z}$, $c=\frac{z-x}{z+x}$, show that

$$(1-a)(1-b)(1-c)=(1+a)(1+b)(1+c).$$

32. If a and b are positive, which is the greater

$$\frac{a+3b}{a+2b} \text{ or } \frac{a+2b}{a+b}\ ?$$

33. Simplify $\frac{1}{a-b}-\frac{1}{2(a+b)}-\frac{a+3b}{2(a^2+b^2)}-\frac{4b^3}{a^4-b^4}$.

34. Add $\frac{x-y}{z^2-(x-y)^2}$, $\frac{y-z}{x^2-(y-z)^2}$, $\frac{z-x}{y^2-(z-x)^2}$.

35. Simplify $\frac{4}{x-4}-\frac{3}{x-3}+\frac{1}{x-1}-\frac{2}{x-2}$.

36 When $x=\frac{ac}{a+c}$, find the value of

$$\frac{x-2a}{x+2c}-\frac{x+2a}{x-2c}+\frac{4ac}{x^2-4c^2}.$$

37. Simplify $\left\{\frac{x+\frac{y-x}{1+xy}}{1-\frac{x(y-x)}{1+xy}}-\frac{y-\frac{y-x}{1-xy}}{1-\frac{y(y-x)}{1-xy}}\right\}\div\left(\frac{y}{x}-\frac{x}{y}\right)$.

CHAPTER XVI

FRACTIONAL EQUATIONS

143. If an equation involves fractions, the fractions may be removed by multiplying every term by the same quantity.

In Chapter VI. simple examples of fractional equations were given.

The case in which two fractions are equal deserves special attention.

Thus, if $\frac{a}{b}=\frac{c}{d}$ and each side is multiplied by bd we have

$$\frac{a}{b}\times bd=\frac{c}{d}\times bd.$$

$$\therefore\quad ad=bc.$$

144. Cross Multiplication. It is thus seen that when two fractions are equal, we can remove the fractions by multiplying the numerator of each fraction by the denominator of the other and equating the results. This operation is sometimes called **cross multiplication.**

Ex. 1.—Solve $\dfrac{x-5}{x-7}=\dfrac{x+3}{x+9}.$

Cross multiply, $(x-5)(x+9)=(x+3)(x-7),$

$$\therefore\quad x^2+4x-45=x^2-4x-21,$$

$$\therefore\quad 8x=24,$$

$$\therefore\quad x=3.$$

Verify by substitution.

This method is applicable only when a single fraction appears on each side of the equation.

Ex. 2.—Solve $\frac{4x+17}{x+3} = 7 - \frac{3x-10}{x-4}$.

Simplify the right-hand member and we have

$$\frac{4x+17}{x+3} = \frac{4x-18}{x-4}.$$

Now cross multiply, complete the solution and verify.

EXERCISE 96

Solve and verify :

1. $\frac{3x+1}{2} = \frac{x+12}{3}$.

2. $\frac{3x-5}{5x-3} = \frac{3x-1}{5x-1}$.

3. $\frac{x}{3} - \frac{x-1}{11} = x - 9$.

4. $\frac{x-2}{x-3} - \frac{x-6}{x-5} = 0$.

5. $\frac{x+2}{5} = \frac{x-2}{2} - \frac{x-1}{7}$.

6. $\frac{x+1}{2} - \frac{x-3}{3} = \frac{x+30}{13}$.

7. $\frac{7x-3}{2} + 1 = \frac{2x+5}{3}$.

8. $\frac{x^2+7x-6}{x^2+5x-10} = \frac{x+1}{x-1}$.

9. $\frac{x+3}{x-1} = 2 - \frac{x+1}{x-3}$.

10. $\frac{1}{x-1} + \frac{1}{x} = \frac{2}{x+1}$.

11. $\frac{1}{x+2} + \frac{2}{x-2} = \frac{3}{x-3}$.

12. $\frac{x-5}{3x-2} - \frac{2x+3}{6x-1} = 0$.

13. $\frac{2x+38}{x+12} = \frac{6x+8}{2x+1} - 1$.

14. $\frac{y-8}{y^2-8y+15} = \frac{y-12}{y^2-12y+30}$

15. $\frac{2x+7}{3} - \frac{8x+19}{12} = \frac{5x+11}{7x+9}$.

16. $\frac{x-1}{x-2} + \frac{x-5}{x-3} = 2$.

17. $\frac{6-8x}{3-x} + \frac{3}{1-x} = 8$.

18. $\frac{2x+7}{x+1} + \frac{3x-5}{x+2} = \frac{5x+9}{x+3}$.

19. $\frac{4x^3+4x^2+8x+1}{2x^2+2x+3} = \frac{2x^2+2x+1}{x+1}$.

20. $\frac{2x-2}{x-3} = \frac{x-4}{x-5} + 1$.

21. $\frac{2-3x}{1\cdot5} + \frac{5x}{1\cdot25} - \frac{x-2}{1\cdot8} = \frac{2x-3}{9} + 3\frac{7}{18}$.

22. Solve $\frac{3x^2-4x+5}{3x-4}=\frac{4x^2-5x+7}{4x-5}$ by first reducing each fraction to a mixed expression.

23. Find three consecutive numbers so that the sum of $\frac{1}{2}$ of the first, $\frac{1}{5}$ of the second and $\frac{1}{6}$ of the third may be 30.

24. Divide 300 into two parts so that if one be divided by 5 and the other by 7, the difference of the quotients will be 18. Give two answers.

25. How much water must be added to 100 lb. of a 4% solution of salt to make a 3% solution ?

26. A pupil was told to add 3 to a number and to divide the result by 5. Instead of doing so he subtracted 3 and multiplied by 5 and obtained the correct answer. What was the number ?

27. A man bought 180 lb. of tea and 560 lb. of coffee, the coffee costing $\frac{8}{15}$ as much as the tea per lb. He sold the tea at a loss of 25% and the coffee at a gain of 50%, and gained $62·60 on the whole. What did the tea cost per lb. ?

28. I sold some butter at 25c. a lb. If I had received 5c. more for 1 lb. less, I would have received 2c. more per lb. How many lb. did I sell ?

29. If I walk to the station at the rate of 11 yards in 5 seconds I have 7 minutes to spare ; if I walk at the rate of 10 yards in 6 seconds I am 3 minutes late. How far is it to the station ?

145. Fractions with similar Denominators.

Ex. 1.—Solve $\frac{x+6}{11}-\frac{2x-18}{3}+\frac{2x+3}{4}=\frac{16}{3}+\frac{3x+4}{12}$.

Here we might multiply each term by the L.C.D., which is 132. It will be found simpler, however, to remove all the fractions but the first, to the same side of the equation, as they are easily reduced to a common denominator.

$$\frac{x+6}{11}=\frac{2x-18}{3}-\frac{2x+3}{4}+\frac{16}{3}+\frac{3x+4}{12}.$$

$$\therefore \quad \frac{x+6}{11}=\frac{4(2x-18)-3(2x+3)+64+3x+4}{12}.$$

Now simplify, cross multiply and complete the solution. The correct answer is $x=5$.

This problem shows that the denominators of certain fractions are such that these fractions can be conveniently combined when they are grouped on one side of the equation.

Ex. 2.—Solve
$$\frac{2x+3}{x+1}=\frac{4x+5}{4x+4}+\frac{5x+4}{5x+1}.$$

Since $4x+4=4(x+1)$, it is seen that it is simpler to combine the second fraction with the first than with the third.

$$\frac{2x+3}{x+1}-\frac{4x+5}{4(x+1)}=\frac{5x+4}{5x+1}.$$

Subtract the first two fractions, complete the solution and verify the result.

Ex. 3.—Solve $\frac{x-1}{x-2}-\frac{x-2}{x-3}-\frac{x-4}{x-5}+\frac{x-5}{x-6}=0.$

Here it is too laborious to multiply all the fractions by the L.C.D. It will be found easier to change the equation so as to have two fractions on each side, then simplify each side and cross multiply.

Solve by transposing the last two fractions, also by transposing the second and fourth, and compare the results.

Ex. 4.—Solve
$$\frac{1}{x+5}+\frac{1}{x+2}=\frac{1}{x+4}+\frac{1}{x+3}.$$

Adding on each side,
$$\frac{2x+7}{x^2+7x+10}=\frac{2x+7}{x^2+7x+12}, \qquad (1)$$

$$\therefore \quad \frac{2x+7}{x^2+7x+10}-\frac{2x+7}{x^2+7x+12}=0,$$

$$\therefore \quad (2x+7)\left(\frac{1}{x^2+7x+10}-\frac{1}{x^2+7x+12}\right)=0,$$

$$\therefore \quad 2x+7=0 \text{ or } \frac{1}{x^2+7x+10}-\frac{1}{x^2+7x+12}=0, \qquad \text{(art. 101)}$$

$$\therefore \quad x=-3\tfrac{1}{2} \text{ or } x^2+7x+10=x^2+7x+12.$$

Since the equation $x^2+7x+10=x^2+7x+12$ is impossible, the only root of the given equation is $x=-3\frac{1}{2}$. (Verify this root.)

If in line (1) we divide each side of the equation by $2x+7$ an impossible equation will result. *It is not allowable to divide both sides of an equation by a common factor unless we know that the factor is not zero.* Here $2x+7$ might be equal to zero, and, in fact, would be if $x=-3\frac{1}{2}$. If $x=-3\frac{1}{2}$ the equation in line (1) is satisfied as each side becomes zero.

Solve the equation by writing it in the form

$$\frac{1}{x+5}-\frac{1}{x+4}=\frac{1}{x+3}-\frac{1}{x+2}.$$

EXERCISE 100

Solve and verify:

1. $\frac{2x+1}{5}-\frac{6x-1}{15}=\frac{3x-2}{6x+3}.$

2. $\frac{2x+3}{4}-\frac{x-1}{6x-8}=\frac{x+2}{2}.$

3. $\frac{x+3}{7}-\frac{3x+5}{6x+2}=\frac{2x+1}{14}.$

4. $\frac{6x+1}{4}-\frac{3x-1}{2}-\frac{2x-1}{3x-2}=0.$

5. $\frac{4}{x-8}+\frac{3}{2x-16}=\frac{29}{24}+\frac{2}{3x-24}.$

6. $\frac{5}{12}+\frac{5x-5}{12x+8}=\frac{6x+7}{9x+6}.$

7. $\frac{13x-10}{36}+\frac{4x+9}{18}-\frac{7x-14}{12}=\frac{23x-88}{17x-66}.$

8. $\frac{3x-4}{6x-9}=\frac{1}{12}-\frac{6x-5}{8x-12}.$

9. $\frac{5x-17}{13-4x}+\frac{2x-11}{14}-\frac{23}{42}=\frac{3x-7}{21}.$

10. $\frac{1}{10}-\frac{5x-7}{10x-5}=\frac{4x-3}{4x-2}.$

11. $\frac{1}{x-1}-\frac{1}{x-2}=\frac{1}{x-3}-\frac{1}{x-4}.$

12. $\frac{1}{x-10}-\frac{1}{x-5}=\frac{1}{x-7}-\frac{1}{x-2}.$

13. $\frac{1}{3x+12}+\frac{1}{6x+24}=\frac{3}{2x+10}-\frac{1}{x+6}.$

14. $\frac{3}{x-5}-\frac{5}{x-7}=\frac{8}{7-x}-\frac{1}{5-x}.$

15. $\frac{x-8}{x-10}+\frac{x-4}{x-6}=\frac{x-5}{x-7}+\frac{x-7}{x-9}$ (transpose terms)

16. $\frac{2x-27}{x-14}+\frac{x-7}{x-8}=\frac{x-12}{x-13}+\frac{2x-17}{x-9}$.

17. Solve the preceding example by changing $\frac{2x-27}{x-14}$ into $2+\frac{1}{x-14}$ and making similar changes in the other fractions.

18. $\frac{4x-17}{x-4}+\frac{10x-13}{2x-3}=\frac{8x-30}{2x-7}+\frac{5x-4}{x-1}$.

19. $\frac{5x-64}{x-13}-\frac{2x-11}{x-6}=\frac{4x-55}{x-14}-\frac{x-6}{x-7}$.

20. $\frac{x-1}{x+1}+\frac{x+1}{x-2}+\frac{x-2}{x-1}=3$.

21. Solve $\frac{x-a}{b+c}+\frac{x-b}{c+a}+\frac{x-c}{a+b}=3$.

22. If $a^2-b^2=a-b$, does it follow that a must be equal to b? What is the alternative conclusion?

146. Literal Equations with one Unknown. Equations often occur in which the known quantities are represented by letters instead of numbers.

These are called **literal equations.**

The same methods are used in solving them as were used in solving equations with numerical coefficients.

Ex. 1.—Solve $ax=bx+c$.	Solve $8x=3x+20$.
$ax-bx=c,$	$8x-3x=20,$
$\therefore\ x(a-b)=c,$	$\therefore\ 5x=20,$
$\therefore\ x=\frac{c}{a-b}$	$\therefore\ x=\frac{20}{5}=4.$

Here the letters a, b, c represent some known numbers whose values, however, are not stated, while x represents the unknown whose value is to be found in terms of a, b and c.

Usually the earlier letters of the alphabet are used to represent known quantities, and the later ones x, y, z to represent unknown ones.

Compare the two solutions given. They are practically identical. When we work with numerical coefficients the result can usually be expressed in a simpler form.

NOTE.—The pupil must not make the mistake of giving $x=\dfrac{bx+c}{a}$ as a solution of $ax=bx+c$.

This statement is true, but it is not a solution, since it does not give the value of the unknown in terms of known quantities only.

Ex. 2.—Solve $a(x-2)-b=a-2x$.

Removing brackets, $ax-2a-b=a-2x$.

Transposing,
$$ax+2x=a+2a+b,$$
$$\therefore\ x(a+2)=3a+b,$$
$$\therefore\ x=\frac{3a+b}{a+2}.$$

The result should be verified by substitution, but this will frequently be found more troublesome than the solution. When it is not verified in the usual way, the pupil should review his work to ensure accuracy.

Ex. 3.—Solve $\dfrac{x-b}{a-x}=\dfrac{x-a}{b-x}$.

Cross multiply, $bx-b^2-x^2+bx=ax-a^2-x^2+ax$,
$$\therefore\ 2bx-2ax=b^2-a^2,$$
$$\therefore\ x=\frac{b^2-a^2}{2(b-a)}=\frac{b+a}{2}.$$

Verify by substitution.

EXERCISE 101

Solve for x, verify 1-12 :

1. $mx+a=b$.
2. $ax=bx+2$.
3. $a+\dfrac{x}{b}=c$.
4. $\dfrac{x+a}{x-a}=\dfrac{7}{3}$.
5. $\dfrac{x-a}{x+b}=\dfrac{a}{b}$.
6. $\dfrac{x-c}{x+c}=\dfrac{a-b}{a+b}$.
7. $\dfrac{a}{x}=\dfrac{b}{x-a+b}$.
8. $\dfrac{x-a}{x-b}=\dfrac{x-b}{x-a}$.
9. $\dfrac{x-a}{2b}+\dfrac{x-b}{2a}=1$.
10. $\dfrac{ax}{b}-\dfrac{c}{d}=\dfrac{b}{a}-\dfrac{dx}{c}$.
11. $a(x-a)+b(x-b)=0$.
12. $\dfrac{1}{a}-\dfrac{1}{x}=\dfrac{1}{x}-\dfrac{1}{b}$.
13.* $(a+x)(b+x)=(c+x)(d+x)$.

14. $(ax-b)(bx+a)=a(bx^2-a)$.   15. $\frac{a(a-x)}{b}-\frac{b(b+x)}{a}=x$.

16. $\frac{a}{x-a}-\frac{b}{x-b}=\frac{a-b}{x-c}$.

17. $x(x-a)+x(x-b)=2(x-a)(x-b)$.

18. $(x-a)(x-b)=(x-a-b)^2$.

19. $\frac{1}{x-a}-\frac{1}{x-2a}=\frac{1}{x-3a}-\frac{1}{x-4a}$.

20 $(x-a)(x-b)-(x+a)(x+b)=(a+b)^2$.

21. $(a+x)(b+x)-a(b+c)=\frac{a^2c}{b}+x^2$.

22. $(a+x)(b-x)+x^2=b(a+x)-\frac{bc^2}{a}$.

23. $a^2x+b^3+abx=a^3-b^2x$.

24. The excess of a number over a is three times its excess over b. Find the number.

25. Divide the number a into two parts so that one part may contain b as often as the other will contain c.

26. Divide a into two parts so that m times the greater may exceed n times the less by b.

27. A rectangle is a feet longer and b feet narrower than a square of the same area. Find the side of the square.

28. If a number be divided by a, the sum of the divisor, quotient and one-third of the number is b. Find the number.

29. A man sells a acres more than the mth part of his farm and has b acres more than the nth part left. How many acres were in the farm ?

30. Solve $(a-x)(b-x)=(c-x)(d-x)$.

Check by putting $a=1$, $b=6$, $c=2$, $d=3$.

31. If $s=\frac{n}{2}(a+l)$, solve for n; for a; for l.

32. If $s=\frac{rl-a}{r-1}$, solve for a; for l; for r.

33. If $s=at+\frac{1}{2}gt^2$, solve for a; for g.

147. Literal Equations with two Unknowns. Every simple equation in x and y may be reduced to the form $ax+by=c$, where a, b, and c represent known quantities.

If two equations in x and y with literal coefficients be given, the equations may be solved by the same methods as were used with equations with numerical coefficients.

Ex. 1.—Solve

$$ax+by=c,$$
$$ax-by=d.$$

Adding, $2ax=c+d, \quad \therefore \; x=\dfrac{c+d}{2a}.$

Subtracting, $2by=c-d, \quad \therefore \; y=\dfrac{c-d}{2b}.$

Verify in the usual way.

Ex. 2.—Solve

$$ax+by=c, \qquad (1)$$
$$mx+ny=k. \qquad (2)$$

Multiply (1) by n, $nax+nby=cn,$

Multiply (2) by b, $bmx+bny=kb.$

Subtracting, $x(na-bm)=cn-kb.$

$$\therefore \; x=\frac{cn-kb}{na-bm}.$$

We might substitute this value of x in either of the given equations to find y, but it is simpler to solve for y in the same way as we did for x.

Eliminate x from the two equations and find $y=\dfrac{cm-ak}{bm-an}$ or $\dfrac{ak-cm}{an-bm}$.

Ex. 3.—Solve

$$a_1x+b_1y=c_1,$$
$$a_2x+b_2y=c_2.$$

Here the symbols a_1, a_2, etc., are used to represent known quantities. They are read "*a one, a two, b one, etc.*" There is no relation in value between a_1 and a_2, nor a_1 and b_1. The notation is used to obviate the necessity of employing many different letter forms.

Solve these equations as in the preceding example and obtain

$$x=\frac{b_2c_1-b_1c_2}{a_1b_2-a_2b_1}, \quad y=\frac{a_2c_1-a_1c_2}{b_1a_2-b_2a_1}.$$

EXERCISE 102

Solve for x and y, verify 1-12:

1. $mx+ny=a,\quad mx-ny=b.$

2. $lx+my=m,\quad mx+ly=l.$

3. $px+qy=r,\quad x+y=0.$

4. $ax+by=a^2+b^2,\quad x+y=a+b.$

5. $ax+by=2ab,\quad bx-ay=b^2-a^2.$

6. $ax+by=2,\quad a^2x-b^2y=a-b.$

7. $ax-by=2a^2+3b^2,\quad bx+ay=-ab.$

8. $ax-by=2ab,\quad 2bx+2ay=3b^2-a^2.$

9. $a^2x+b^2y=a^2-ab+b^2,\quad ax-by=a-b.$

10. $\frac{x}{a}+\frac{y}{b}=3,\quad \frac{x}{a}-\frac{y}{b}=1.$

11. $\frac{x}{a}+\frac{y}{b}=2,\quad ax-by=a^2-b^2.$

12. $a^2x-b^2y=a^3+b^3,\quad \frac{x}{a}-\frac{y}{b}=2.$

13.* $a^2x+b^2y=a^3-b^3,\quad bx+ay=0.$

14. $a_1x+b_1y=c_1,\quad x+y=1.$

15. $\frac{a}{x}+\frac{b}{y}=-\frac{1}{2},\quad \frac{3a}{x}-\frac{4b}{y}=5\frac{1}{2}.$

16. $(a+b)x-(a-b)y=a^2+b^2,\quad x-y=a-b.$

17. $\frac{a_1}{x}+\frac{b_1}{y}=c_1,\quad \frac{a_2}{x}+\frac{b_2}{y}=c_2.$

18. $x+\frac{ay}{a-b}=\frac{ax}{a+b}+y=b.$

19. $\frac{1\cdot5a}{x}+\frac{3\cdot2b}{y}=22,\quad \frac{\cdot5a}{x}-\frac{\cdot35b}{y}=\cdot25.$

20. If $ax+by=c$ and $x-y=1$, prove that $x(c-a)=y(b+c)$.

21. If $y=ax+b$ and $x=py-q$, prove that $y(q-bp)=x(aq-b)$.

22. If $bx+ay+cz=ac+bc$, $ax+by=ac$, $cy+dz=ad$, solve for x, y and z.

23. What is the value of m, in terms of a and b, if the following equations are consistent

$$ax+3by=a^2+3b^2,$$
$$3x+y=3a+b,$$
$$4x-3y=m\ ?$$

EXERCISE 108 (Review of Chapter XVI)

Solve and verify 1-24:

1. $[illegible](7-x)-[illegible](11-x)=[illegible](x-8)$.

2. $\frac{3x-2}{2x-3}-\frac{x+17}{x+10}=\frac{1}{2}$.

3. $\frac{x}{2}+\frac{y}{3}=[illegible],\ \frac{x}{3}+\frac{y}{2}=[illegible]$.

4. $\frac{2x+7}{7}-\frac{9x-8}{11}=\frac{x-11}{2}$.

5. $\frac{2x-3}{x-4}+\frac{3x-2}{x-8}=\frac{5x^2-29x-4}{x^2-12x+32}$.

6. $\frac{x+1}{2}-\frac{3}{x}=\frac{x}{3}-\frac{5-x}{6}$.

7. $\frac{3x-2}{7}-\frac{2(4x+1)}{3}+\frac{5x-2}{2}=5[illegible]$.

8. $\frac{3x-7}{20}-(x-20\frac{1}{3})=1$.

9. $\frac{5+3x}{2}-\frac{4x-7}{3}=\frac{16x-27}{21}-\frac{x+3}{5}$.

10. $\frac{2x-3}{2x+1}+\frac{3x-7}{3x+5}=2$.

11. $a(x-a)-b(x-b)=(a+b)(x-a-b)$.

12. $ax+b=bx+a$.

13. $\frac{8x+5}{10}-\frac{9x-3}{7x+2}=\frac{4x-3}{5}$.

14. $\frac{4-x}{13}-\frac{2+4x}{7}=6-x$.

15. $\frac{x}{a}+\frac{y}{b}=1,\ \frac{x}{3a}-\frac{y}{4b}=5$.

16. $\frac{ax+b}{ax-b}=\frac{cx+d}{cx-d}$.

17. $\frac{5}{3x}+\frac{2}{5y}=7,\ \frac{7}{6x}+\frac{1}{10y}=3$.

18. $\frac{x}{a-2b}=\frac{1}{2}+\frac{2x}{2a-b}$.

19. $ax+by=2ab,\ ay-bx=a^2-b^2$.

20. $\frac{x}{x+b-a}=1-\frac{b}{x+b-c}$.

21. $\frac{x+2}{x+3}-\frac{x+3}{x+4}=\frac{x+5}{x+6}-\frac{x+6}{x+7}$.

22. $\frac{3}{x+10}+\frac{2}{x-10}=\frac{5}{x-2}$.

23. $\frac{x-1}{x-2}-\frac{x-3}{x-4}-\frac{x-5}{x-6}+\frac{x-7}{x-8}=0$.

24. $\frac{x+1}{3x-4}=\frac{1}{5}+\frac{8x-3}{15x-20}$.

25.* $(3a-x)(a-b)+2ax=4b(a+x)$.

26. $\frac{3x}{a}+2b(a-c)+\frac{x}{b}=c(a+b)+\frac{2x}{c}$.

27. $2x+\frac{2x-3y}{5}=\frac{y+3}{4}+7,\ \frac{2y+1}{2}-\frac{8-x}{3}=24\frac{1}{2}-4y$.

28. $\frac{y}{3}+\frac{2y-3x}{6}=8,\ \frac{7x-3y}{2}-x=11$.

29. $\frac{4}{x}+\frac{3}{y}=-7,\ 3x-2y=-10xy$.

30. $\frac{2y}{5}+\frac{5x}{3}=2\frac{5}{6}xy,\ 5y-2x=24xy.$

31. What value of y will make

$$\frac{x+5}{3}+\frac{y-1}{4} \text{ equal to } \frac{x-1}{3}+\frac{y+11}{8}\,?$$

32. Two sums of money are together equal to \$1000, and 5½% of the larger exceeds 6½% of the smaller by 16 cents. Find the sums.

33. Find a fraction such that if 4 be added to its numerator it becomes equal to ¾, but if 4 be added to its denominator it becomes ½.

34. If $ax+b=cx+d$, give the argument which leads to the conclusion that $x=\frac{d-b}{a-c}$, indicating at what point it is assumed that a and c are unequal.

35. Take any two proper fractions whose sum is unity. Add unity to the difference between their squares. Show that the result is always twice the greater fraction.

36. A man has \$30,000 invested, part at 4¾% and the rest at 5¼%. He receives \$65 per annum more income from the former than from the latter. How much is invested at each rate ?

37. The sum of three numbers a, b, c is 3036; a is the same multiple of 7 that b is of 4, and also the same multiple of 5 that c is of 2. Find the numbers.

38. If $s=\frac{n}{2}(2a+nd-d)$, solve for a; for d.

39. If $ax-by=a^2+b^2$, $x-y=2b$ and $x^2+y^2=c$, find c in terms of a and b.

40. A man can walk 2½ miles an hour up hill and 3½ miles per hour down hill. He walks 56 miles in 20 hours on a road no part of which is level. How much of it is up hill ?

41. A farm cost 3¾ times as much as a house. By selling the farm at 7½% gain and the house at 10% loss, \$2754 was received. Find the cost of each.

42. In 10 years the total population of a city increased 11%. The foreign population, which was originally $\frac{1}{11}$ of the total, decreased by 1160 and the native population increased by 12%. Find the total population at the end of the period.

CHAPTER XVII

EXTRACTION OF ROOTS

148. Square Roots by Inspection. In art. 65 we have seen that the square root of any trinomial, which is a perfect square, may be written down by inspection.

We have also seen that every quantity has two square roots differing only in sign.

Thus, the square root of $a^2+2ab+b^2$ is $\pm(a+b)$,
and of $a^2-2ab+b^2$ is $\pm(a-b)$.

$$\pm(a+b)=a+b \text{ or } -a-b\,; \quad \pm(a-b)=a-b \text{ or } b-a.$$

If we had written $a^2-2ab+b^2$ in its equivalent form $b^2-2ab+a^2$, it is seen that $b-a$ is a root.

It is usual, however, to give only the square root which has its first term positive, and we say that the square root of $a^2+2ab+b^2$ is $a+b$ and of $a^2-2ab+b^2$ is $a-b$ or $b-a$.

EXERCISE 104 (Oral)

State the square of:

1. $-abc$.
2. $x+1$.
3. $-x-1$.
4. $2a-3b$.
5. a^2+1.
6. x^2-x.
7. $a+b+c$.
8. $a+b-1$.
9. $2a+b-c$

State the square root of:

10. $16x^2y^2$.
11. $\frac{1}{4}a^2b^4$.
12. $x^2+2ax+a^2$.
13. a^2-2a+1.
14. $4a^2-12ab+9b^2$.
15. $9x^2-30xy+25y^2$.
16. $x^2+x+\frac{1}{4}$.
17. $a^4+2a^3+a^2$.
18. $16x^4-48x^2+36$.
19. $x^2+y^2+z^2+2xy+2xz+2yz$.
20. $a^2+b^2+c^2-2ab-2ac+2bc$.
21. $4a^2+9b^2+1+12ab-4a-6b$.

149. Formal Method of Finding Square Root. When the square root of an expression of more than three terms is required, it is not always possible to write down the square root by inspection.

Thus, to find the square root of

$$9x^4-12x^3+10x^2-4x+1.$$

Here we could say that the first term in the square root is $3x^2$, and that the last term is either $+1$ or -1, but it is evident that there must be another term as well.

Let us again examine the square of the binomial $a+b$, which is $a^2+2ab+b^2$.

The first term of the square root is a, which is the square root of a^2. The second term of the square root, b, may be obtained in two different ways, either from the last term, b^2, or from the middle term, $2ab$.

Let us now see how we could obtain the second term in the square root from the middle term $2ab$. This term is twice the product of a which is already found, and of the last term of the square root which is still to be found.

If twice the product of a and the last term is $2ab$, then we can find the last term of the root by dividing $2ab$ by $2a$, which gives b.

The quantity $2a$ which we use to find the second term in the square root is called the **trial divisor.**

Since $a^2+2ab+b^2=a^2+b(2a+b)$, we see that the **complete divisor** is $2a+b$, that is, the trial divisor with the second term in the square root added to it.

$$\begin{array}{r|l}
 & a^2+2ab+b^2 \;\big|\; \underline{a+b} \\
 & a^2 \\
\hline
\underline{2a+b} & +2ab+b^2 \\
 & \underline{2ab+b^2}
\end{array}$$

The steps in the process are :

(1) The square root of a^2 is a. The square of a is subtracted from the expression leaving $2ab+b^2$.

(2) The trial divisor for obtaining the second term in the square root is $2a$. When $2a$ is divided into $2ab$ the quotient is b, the second term in the root.

(3) The complete divisor is $2a+b$, and when this is multiplied by b

and the product subtracted from $2ab+b^2$ there is no remainder. The square root is then $a+b$.

It might be thought that step (3) is unnecessary, as the root has already been found in (1) and (2). It is unnecessary if we take for granted that the expression is a perfect square.

If you attempt to find the square root of $a^2+2ab+4b^2$ and do not go beyond steps (1) and (2), you would get the result $a+b$, as before. This, however, is not the correct result. Why?

We can now extend the method to find the square root of a quantity of more than three terms.

$$\begin{array}{r|l}
 & 9x^4-12x^3+10x^2-4x+1 \,\underline{|\,3x^2-2x+1} \\
 & 9x^4 \\
\cline{2-2}
\underline{6x^2-2x} & -12x^3+10x^2-4x+1 \\
 & -12x^3+\ \ 4x^2 \\
\cline{2-2}
\underline{6x^2-4x+1} & 6x^2-4x+1 \\
 & 6x^2-4x+1 \\
\cline{2-2}
\end{array}$$

After finding the first two terms in the root, as in the previous example, the $3x^2-2x$ is treated as a single quantity and the second trial divisor is twice $3x^2-2x$ or $6x^2-4x$.

The square root is $3x^2-2x+1$.

150. Verifying Square Root. We might verify the result in the preceding example by writing down the square of $3x^2-2x+1$. Verify in this way.

A simple method of checking is to substitute a particular number for x.

When $x=1$, $9x^4-12x^3+10x^2-4x+1=9-12+10-4+1=4$,
and $3x^2-2x+1=3-2+1=2$.
Since the square root of 4 is 2, we presume the work is correct.

EXERCISE 105

Find the square root, by the formal method, and verify the results:

1. $x^2+12x+36$.	2. $9a^2-6a+1$.
3. $9x^2+24xy+16y^2$.	4. $25x^2-10xy+y^2$.
5. $1-18ab+81a^2b^2$.	6. $49a^4-28a^2b^2+4b^4$.
7. $a^4+2a^3-3a^2-4a+4$.	8. $4x^4+4x^3+5x^2+2x+1$.

9. $x^4-6x^3+17x^2-24x+16$. **10.** $9a^4-12a^3+34a^2-20a+25$

11. $a^4-4a^3b+6a^2b^2-4ab^3+b^4$. **12.** a^4-4a^3+8a+4.

13. $9a^4+12a^3b+34a^2b^2+20ab^3+25b^4$.

14. $x^6-4x^5+6x^3+8x^2+4x+1$.

15. $x^4-2x^3+2x^2-x+\frac{1}{4}$.

16. $\frac{a^4}{b^4}+\frac{4a^3}{b^3}+\frac{2a^2}{b^2}-\frac{4a}{b}+1$.

17. $a^2-4ab+6ac+4b^2-12bc+9c^2$.

18.* Simplify $a(a+1)(a+2)(a+3)+1$, and find its square root.

19. By extracting the square root of $x^4+4x^3+6x^2+3x+7$, find a value of x which will make it a perfect square. (Verify by substitution.)

20. If the square root of $x^4-8x^3+30x^2-56x+49$ be x^2+mx+7, what is the value of m?

21. Using factors, find the square root of

$$(x^2+3x+2)(x^2+5x+6)(x^2+4x+3).$$

22. Find the first three terms in the square root of $1-2x-3x^2$ and of $4-12x$.

23. When $x=10$, the number 44,944 may be written

$$4x^4+4x^3+9x^2+4x+4.$$

Find the square root of the latter and thus deduce the square root of 44,944.

151. In algebra, an expression of which the square root is required is usually a perfect square. When such is the case the formal method may be greatly abbreviated.

EX. 1.—Find the square root of

$$x^4-4x^3+10x^2-12x+9.$$

The first term is x^2 and the last is $+3$ or -3.

The trial divisor for obtaining the second term of the root is $2x^2$, therefore the second term is $-4x^3 \div 2x^2$ or $-2x$.

$\therefore$ the square root is x^2-2x+3 or x^2-2x-3.

If we square x^2-2x+3, the term containing x will be twice the product of $-2x$ and 3 or $-12x$. If we square x^2-2x-3, the term containing x will be $+12x$.

We thus see that if the expression is a perfect square, the square root is x^2-2x+3.

Check this by putting $x=1$.

Ex. 2.—Find the square root of

$$4x^4+20x^3+13x^2-30x+9.$$

What is the first term in the square root ? What is the trial divisor ? What is the second term in the root ? What may the last term be ? What is the square root ? (Verify your answer.)

Ex. 3.—Find the square root of

$$-3a^3+\tfrac{25}{9}+a^4-5a+\tfrac{67}{12}a^2.$$

Write the expression in descending powers of a.

$$a^4-3a^3+\tfrac{67}{12}a^2-5a+\tfrac{25}{9}.$$

The first term in the root is a^2. The trial divisor is $2a^2$, therefore the second term is $-3a^3\div 2a^2$ or $-\tfrac{3}{2}a$.

$\therefore$ the root is $a^2-\tfrac{3}{2}a+\tfrac{5}{3}$ or $a^2-\tfrac{3}{2}a-\tfrac{5}{3}$.

Which is it ? (Verify by squaring.)

Ex. 4.—Find the square root of

$$4a^2-4a+9-\frac{4}{a}+\frac{4}{a^2}.$$

Here the terms are already arranged in descending powers of a, the term $+9$ coming between a and $\dfrac{1}{a}$.

The first term in the root is $2a$, the second is $-4a\div 4a$ or -1, and the last is $\pm\dfrac{2}{a}$.

Complete and verify.

It will be recognized that it is only in the most complicated cases that it is necessary to use the formal method in full. It is advisable to use the contracted method whenever possible.

EXERCISE 106

Find the square root, using any method you prefer. Verify the results.

1. $x^4+2x^3-x^2-2x+1$.

2. $x^4-4x^3+6x^2-4x+1$.

3. $a^4-6a^3+5a^2+12a+4$.

4. $x^4+8x^3+12x^2-16x+4$.

5. $9a^4-6a^3+13a^2-4a+4$.

6. $x^4+6x^3y+7x^2y^2-6xy^3+y^4$

7. $4x^4+20x^3-3x^2-70x+49$.

8. $1-10x+27x^2-10x^3+x^4$.

9. $67x^2+49+9x^4-70x-30x^3$.

10. $a^{12}-8a^9+18a^6-8a^3+1$.

11. $x^4+2x^3-x+\frac{1}{4}$.

12. $x^4-2x^3+\frac{3}{2}x^2-\frac{1}{2}x+\frac{1}{16}$.

13. $a^4-6a^2+11-\dfrac{6}{a^2}+\dfrac{1}{a^4}$.

14. $\dfrac{4x^4}{y^4}-\dfrac{4x^3}{y^3}-\dfrac{3x^2}{y^2}+\dfrac{2x}{y}+1$.

15. $\dfrac{x^4}{4}-3x^3+\frac{28}{3}x^2-2x+\frac{1}{9}$.

16. $\dfrac{9a^4}{25}+\dfrac{4a^3}{5}+\dfrac{74a^2}{45}+\dfrac{4a}{3}+1$.

17. $5+\dfrac{4}{x^2}-\dfrac{4}{x}-2x+x^2$.

18. $\dfrac{x^2}{y^2}-\dfrac{2x}{y}+3-\dfrac{2y}{x}+\dfrac{y^2}{x^2}$.

19.* $(x+y)^4-4(x+y)^3+6(x+y)^2-4(x+y)+1$.

20. $x^2(x-5a)(x-a)+a^2(3x-a)^2-3a^2x^2$.

21. $(a-b)^2\{(a-b)^2-2(a^2+b^2)\}+2(a^4+b^4)$.

22. $(a+b)^4-2(a^2+b^2)(a+b)^2+2(a^4+b^4)$.

23. $\left(x^4+\dfrac{1}{x^4}\right)+4\left(x^2+\dfrac{1}{x^2}\right)+6$.

24. If $x^4+6x^3+7x^2+ax+1$ is a perfect square, what is the value of a?

25. If the sum of the squares of any two consecutive integers be added to the square of their product, prove that the result will be a square.

26. If $4x^4+12x^3y+kx^2y^2+6xy^3+y^4$ is a perfect square, find k.

27. If $m=x-\dfrac{1}{x}$ and $n=y-\dfrac{1}{y}$, show that

$$mn+\sqrt{(m^2+4)(n^2+4)}=2xy+\frac{2}{xy}.$$

28. Find the square root of $4x^4+8x^3+8x^2+4x+1$. Check when $x=10$.

152. Cube of a Monomial. When three equal factors are multiplied together, the product is called the **cube** of each of the factors.

Thus, the cube of $2a$ or $(2a)^3 = 2a \,.\, 2a \,.\, 2a = 8a^3$,
the cube of a^2 or $(a^2)^3 \quad = a^2 \,.\, a^2 \,.\, a^2 = a^6$,
and the cube of $3a^3$ or $(3a^3)^3 \quad = 3a^3 \,.\, 3a^3 \,.\, 3a^3 = 27a^9$.

The cube of a monomial is found by writing down the cube of each factor of it.

Thus, the cube of $5ab^2x$ is $125a^3b^6x^3$.

153. Cube of a Binomial. Find the cube of $a+b$ by multiplying its square by $a+b$. Find also the cube of $a-b$.

$$\begin{array}{l} a^2+2ab+b^2 \\ a\ +b \\ \hline a^3+2a^2b+\ ab^2 \\ \quad\ +\ a^2b+2ab^2+b^3 \\ \hline a^3+3a^2b+3ab^2+b^3 \end{array}$$

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3.$$
$$(a-b)^3 = a^3-3a^2b+3ab^2-b^3.$$

Note that in each case the cube contains four terms, in descending powers of a and ascending powers of b, and the numerical coefficients are 1, 3, 3, 1.

The cube of $a-b$ is the same as the cube of $a+b$, except that the signs are alternately plus and minus.

From the forms of these two cubes, the cubes of other expressions may be written down.

Ex. 1. $(x+2y)^3 = x^3+3x^2(2y)+3x(2y)^2+(2y)^3$,
$\qquad\qquad = x^3+6x^2y+12xy^2+8y^3$.

Ex. 2. $(3x-2y)^3 = (3x)^3-3(3x)^2(2y)+3(3x)(2y)^2-(2y)^3$,
$\qquad\qquad = 27x^3-54x^2y+36xy^2-8y^3$.

Ex. 3. $(a+b+c)^3 = (\overline{a+b}+c)^3$,

$\qquad\qquad = (a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3$,
$= a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3$,
$= a^3+b^3+c^3+3(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2)+6abc$.

EXERCISE 107 (1-12, Oral)

Find the cube of:

1. $-\frac{1}{3}$.	2. $-2a$.	3. $-3ab^2$.	4. $-x^2yz^3$.
5. $x+y$.	6. $x-y$.	7. $m+n$.	8. $p-q$.
9. $x+1$.	10. $x-1$.	11. a^2+b.	12. $1-a$.
13. $x+3$.	14. $2x-y$.	15. $2a+3b$.	16. $1-2a$.
17. $a-4b$.	18. $1-a^2$.	19. $a+b-c$.	20. $a-b-c$.

21.* Simplify $(a+b)^3+(a-b)^3$ and $(a+b)^3-(a-b)^3$.

22. Show that $(x+y)^3=x^3+y^3+3xy(x+y)$ and write a similar form for $(x-y)^3$.

23. Simplify $(a+b+c)^3+(a+b-c)^3$.

24. Show that $(a-b)^3+(b-c)^3+(c-a)^3=3(a-b)(b-c)(c-a)$.

25. Show that the difference of the cubes of any two consecutive integers is greater than three times their product by unity.

26. When $x=y+z$, show that $x^3-y^3-z^3=3xyz$.

27. Two numbers differ by 3. By how much does the difference of their cubes exceed nine times their product?

28. Three consecutive integers are multiplied together and the middle integer is added to the product. Show that the result must be the cube of this middle integer. What is the cube root of

$$241\times242\times243+242?$$

154. Cube Root of a Monomial. The cube root of any quantity is one of the three equal factors which were multiplied to produce that quantity.

Thus, the cube root of 8 is 2, of a^3 is a, of $8x^3$ is $2x$, of a^6 is a^2, of $27a^3b^6$ is $3ab^2$.

The cube root of any power of a letter is obtained by dividing the index of the power by 3.

The symbol indicating cube root is $\sqrt[3]{}$.

Thus, $\sqrt[3]{125}=5$, $\sqrt[3]{a^9}=a^3$, $\sqrt[3]{8x^6y^3}=2x^2$

155. Cube Root of a Compound Expression.

The cube root of $a^3+3a^2b+3ab^2+b^3$ is $a+b$,
and of $a^3-3a^2b+3ab^2-b^3$ is $a-b$.

Therefore, when an expression of four terms is known to be a perfect cube, its cube root can at once be written down by finding the cube roots of its first and last terms.

Ex. 1.—The cube root of $x^3-6x^2y+12xy^2-8y^3$ is $x-2y$, since the cube root of x^3 is x and of $-8y^3$ is $-2y$.

Ex. 2.—The cube root of $a^3-2a^2b+\frac{4}{3}ab^2-\frac{8}{27}b^3$ is evidently $a-\frac{2}{3}b$.

In the cube of $a+b$, the second term is $3a^2b$. After finding the first term a of the cube root, we might have found the second term of the root by dividing $3a^2b$ by $3a^2$, that is, by three times the square of the term already found.

Thus, the second term of the cube root in Ex. 1 is

$-6x^2y \div 3x^2$ or $-2y$,

and in Ex. 2 is $-2a^2b \div 3a^2$ or $-\frac{2}{3}b$.

Here *three times the square of the first term* of the root is the trial divisor, corresponding to *twice the first term* in finding the square root.

Ex. 3.—Find the cube root of

$$8x^6+12x^5-30x^4-35x^3+45x^2+27x-27.$$

The first term in the root is $2x^2$ and the last is -3.

The trial divisor for finding the second term of the root is $3(2x^2)$ or $12x^4$.

∴ the second term of the root is $12x^5 \div 12x^4$ or x.

∴ the cube root is $2x^2+x-3$.

It is thus seen that it is easier to find cube root by inspection than to find square root, as in finding cube root there is no ambiguity as to the sign of the last term in the root.

156. Higher Roots. Since $(x^2)^2=x^4$, we may find the fourth root by taking the square root and then the square root of the result.

Also, since $(x^2)^3=x^6$ and $(x^3)^2=x^6$, we can find the sixth root by taking the square root of the cube root, or the cube root of the square root.

Thus, the square root of $x^4+8x^3+24x^2+32x+16$ is x^2+4x+4, therefore the fourth root is $x+2$.

The cube root of $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$ is x^2-2x+1, therefore the sixth root is $x-1$.

EXERCISE 108 (1-15, Oral)

State the cube root of:

1. -64.
2. $27a^3$.
3. $-125a^3b^3$.
4. $-8(a-b)^3$
5. x^3+3x^2+3x+1.
6. $x^3-3x^2y+3xy^2-y^3$.
7. $a^3+6a^2+12a+8$.
8. $8x^3-12x^2+6x-1$.
9. $x^3y^3+3x^2y^2+3xy+1$.
10. $64a^3-144a^2+108a-27$.
11. $125x^3-75x^2+15x-1$.
12. $27x^3-27x^2y+9xy^2-y^3$.
13. $\frac{x^3}{8}-\frac{3x^2}{4}+\frac{3x}{2}-1$.
14. $\frac{x^3}{27}+\frac{2x^2}{3}+4x+8$.
15. $m^3-9m+\frac{27}{m}-\frac{27}{m^3}$.
16. $\frac{x^6}{y^3}-6x^4+12x^2y^3-8y^6$.

17.* In finding the cube root of $x^6+3x^5+6x^4+7x^3+6x^2+3x+1$, what is the first term in the root? What is the last term? What is the trial divisor for finding the second term? What is the cube root? Check by substituting $x=1$.

Find the cube root and check:

18. $1-6x+21x^2-44x^3+63x^4-54x^5+27x^6$.
19. $\frac{x^3}{27}-\frac{x^2}{3}+2x-7+\frac{18}{x}-\frac{27}{x^2}+\frac{27}{x^3}$.
20. $27a^6-108a^5+171a^4-136a^3+57a^2-12a+1$.
21. $(1+3x^2)^3-x^2(3+x^2)^2$.
22. For what value of x will $x^3+3cx^2+2c^2x+5c^3$ be a perfect cube?
23. Find the fourth root of $x^4-4x^3+6x^2-4x+1$.

24. Find the fourth root of $a^4-12a^3+54a^2-108a+81$

25. Find the sixth root of

$$x^6-12x^5+60x^4-160x^3+240x^2-192x+64.$$

EXERCISE 109 (Review of Chapter XVII)

Find the square root of :

1.* $9x^4-24x^3y+28x^2y^2-16xy^3+4y^4$.

2. $x^6+4x^5-2x^4-10x^3+13x^2-6x+1$.

3. $x^{12}+6x^{10}+5x^8-8x^6+16x^4-8x^2+4$.

4. $\frac{1}{4}x^4-\frac{1}{3}x^3+\frac{10}{9}x^2-\frac{2}{3}x+1$.

5. $12a^3x-26a^2x^2+25x^4+9a^4-20ax^3$.

6. $4x^2(7+x^2+3a)+(3a+7)^2$.

7. $(x^2+5x+6)(x^2+7x+12)(x^2+6x+8)$.

8. $(2x^2-x-3)(x^2-4x-5)(2x^2-13x+15)$.

9. $4x^4-20x^3+33x^2-32x+34-\dfrac{12}{x}+\dfrac{9}{x^2}$.

What is the cube root of :

10. $27-135x+225x^2-125x^3$.

11. $8x^6-12x^5+18x^4-13x^3+9x^2-3x+1$.

12. $(a-b)^3+3b(a-b)^2+3b^2(a-b)+b^3$.

13. Find to three terms, the square roots of :

$$1-2x,\ 1-a,\ 4+x.$$

14. Find the value of y for which $x^2-2(a-y)x+y^2$ is a complete square and prove by trial that your result is correct.

15. The first two terms of a perfect square are $49x^4-28x^3$, and the last two are $+6x+\frac{9}{4}$. What must the square root be ?

16. Prove that the product of any four consecutive integers increased by unity is a perfect square.

17. Find the square root of $a^4+4a^3+6a^2+4a+1$ and deduce the square root of 14,641.

18. By finding the cube root, simplify

$$(a+b)^3+3(a+b)^2(a-b)+3(a+b)(a-b)^2+(a-b)^3.$$

19. If $a=b+1$, show that $a^3-b^3-1=3ab$.

20. Show that the product of any four consecutive even integers increased by 16 is a perfect square. How might the result be deduced from No. 16?

21. By inspection, find the values of

$$(a-b)^2+(b-c)^2+(c-a)^2+2(a-b)(b-c)+2(b-c)(c-a)+2(c-a)(a-b),$$
$$(2x-y)^3-3(2x-y)^2(2x+y)+3(2x-y)(2x+y)^2-(2x+y)^3.$$

22. To the square of the double product of any two consecutive integers, add the square of their sum. Prove that the result is always a perfect square.

23. Express in symbols: The difference of the cubes of any two numbers exceeds the cube of their difference by three times their product multiplied by their difference. Prove that this is true.

24. The expression

$$8x^9-36x^8+66x^7-87x^6+105x^5-87x^4+61x^3-42x^2+12x-8$$

is a perfect cube. Find its cube root by getting two terms from the first two terms of the expression and the other two from the last two terms. Check when $x=1$.

25. What number must be added to the product of any four consecutive odd integers so that the sum may be a perfect square?

26. Show that the sum of the cubes of three consecutive integers exceeds three times their product by nine times the middle integer.

27. Find the cube root of

$$(4x-1)^3+(2x-3)^3+6(4x-1)(2x-3)(3x-2).$$

[Note that it is of the form $a^3+b^3+3ab(a+b)$.]

28. If $4x^6+12x^5+5x^4-2x^3$ are the first four terms of an exact square, find the remaining three terms.

CHAPTER XVIII

QUADRATIC SURDS

157. Surd. When the root of a number cannot be exactly found, that root is called a **surd.**

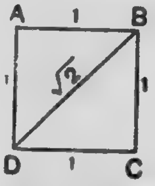
Thus, we cannot find exactly the number whose square is equal to 2, and we represent the number by the symbol $\sqrt{2}$ and we call $\sqrt{2}$ a surd.

If no surd appears in any quantity, it is called a **rational** quantity.

By the arithmetical process of extracting the square root of 2, we can obtain the value of $\sqrt{2}$ to as many decimal places as we please, but its exact value can not be found.

To four decimal places the value of $\sqrt{2}$ is 1·4142. Find the square of 1·4142 by multiplication and see how closely it approximates to 2.

We can find geometrically a line whose length is $\sqrt{2}$ units. In this square, whose side is 1 unit, draw the diagonal BD.



Then, from geometry, we know that

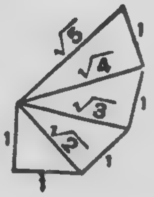
$$BD^2 = AD^2 + AB^2,$$

$$\therefore\ BD^2 = 1^2 + 1^2 = 2,$$

$$\therefore\ BD = \sqrt{2}.$$

On squared paper mark the corners of a square whose side is 10 units. Measure the diagonal and thus estimate as closely as you can the value of $\sqrt{2}$.

Make a diagram like this to show how to represent graphically lines whose lengths are $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$, etc. Take the unit line 1 inch in length. What test have you of the accuracy of your drawing?



158. Quadratic surd. A surd like $\sqrt{2}$ in which the square root is to be found is called a **quadratic surd.** In this Chapter quadratic surds only are considered.

159. Multiplication of Simple Surds.

Since $\sqrt{2}$ represents a quantity whose square is 2,

$$\therefore\ \sqrt{2}\times\sqrt{2}=2=\sqrt{4},$$

also

$$\sqrt{4}\times\sqrt{9}=\sqrt{36}, \text{ because } 2\times 3=6.$$

Similarly, we might expect that $\sqrt{2}\times\sqrt{3}=\sqrt{6}$. That this is true may be shown by finding the square of $\sqrt{2}\times\sqrt{3}$.

$$\begin{aligned}(\sqrt{2}\times\sqrt{3})^2&=\sqrt{2}\times\sqrt{3}\times\sqrt{2}\times\sqrt{3}, \text{ [Just as } (ab)^2=a\,.\,b\,.\,a\,.\,b.\text{]}\\ &=\sqrt{2}\times\sqrt{2}\times\sqrt{3}\times\sqrt{3},\\ &=2\times 3=6.\end{aligned}$$

$$\therefore\ \sqrt{2}\times\sqrt{3}=\sqrt{6}.$$

Similarly,

$$\sqrt{3}\times\sqrt{5}=\sqrt{15},$$

and

$$\sqrt{a}\times\sqrt{b}=\sqrt{ab}.$$

Therefore, *the product of the square roots of two numbers is equal to the square root of the product of the numbers.*

Since $\sqrt{ab}=\sqrt{a}\times\sqrt{b}$, $\therefore\ \sqrt{12}=\sqrt{4}\times\sqrt{3}=2\sqrt{3}$, and

$$\sqrt{50}=\sqrt{25}\times\sqrt{2}=5\sqrt{2};\quad \sqrt{18a^3}=\sqrt{9a^2}\times\sqrt{2a}=3a\sqrt{2a}.$$

Thus, we see that *if there is a square factor under the radical sign, that factor may be removed if its square root be taken.*

Conversely,

$$\begin{aligned}5\sqrt{3}&=\sqrt{25}\times\sqrt{3}=\sqrt{75},\\ a\sqrt{b}&=\sqrt{a^2}\times\sqrt{b}=\sqrt{a^2b},\\ ax\sqrt{my}&=\sqrt{a^2x^2}\times\sqrt{my}=\sqrt{a^2x^2my}.\end{aligned}$$

160. Mixed and Entire Surds. When a surd quantity is the product of a rational quantity and a surd, it is called a **mixed surd.** If there is no rational factor it is called an **entire surd.**

Thus, $5\sqrt{3}$, $a\sqrt{b}$, $(a-b)\sqrt{x-y}$ are mixed surds, and $\sqrt{3}$, $\sqrt{50}$, $\sqrt{ax+b}$ are entire surds.

In the preceding article we have shown that a mixed surd can always be changed into an entire surd, and an entire surd can sometimes be changed into a mixed surd.

A surd is said to be in its simplest form when the quantity under the radical sign is integral and contains no square factor.

Thus, the simplest form of $\sqrt{50}$ is $5\sqrt{2}$.

EXERCISE 110 (1-29, Oral)

Find the product of :

1. $\sqrt{2}, \sqrt{3}$.
2. $\sqrt{5}, \sqrt{5}$.
3. $\sqrt{2}, \sqrt{8}$.
4. $3\sqrt{7}, 2\sqrt{7}$.
5. $\sqrt{3}, \sqrt{5}, \sqrt{2}$.
6. $\sqrt{8}, \sqrt{\frac{1}{2}}$.
7. $\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{3}}, \sqrt{\frac{1}{6}}$.
8. $(\sqrt{abc})^2$.

Express as entire surds :

9. $2\sqrt{3}$.
10. $3\sqrt{2}$.
11. $5\sqrt{5}$.
12. $a\sqrt{b}$.
13. $3a\sqrt{\frac{1}{3}}$.
14. $b\sqrt{\frac{a}{b}}$.
15. $3\sqrt{a-b}$.
16. $(a+b)\sqrt{\frac{a-b}{a+b}}$.

Simplify, by removing the square factor :

17. $\sqrt{8}$.
18. $\sqrt{12}$.
19. $\sqrt{20}$.
20. $\sqrt{75}$.
21. $\sqrt{27}$.
22. $\sqrt{56}$.
23. $\sqrt{162}$.
24. $\sqrt{2a^2}$.
25. $\sqrt{1000x^2}$.
26. $\frac{1}{4}\sqrt{32}$.
27. $\sqrt{(a-b)^2}$.
28. $\frac{1}{a}\sqrt{a^2b}$.

29. Solve $x^2=2$; $3x^2=27$; $\frac{1}{2}x^2=9$.

30. Show by squaring that

$$\sqrt{3}\times\sqrt{7}=\sqrt{21} \text{ and } \sqrt{a}\times\sqrt{b}\times\sqrt{c}=\sqrt{abc}.$$

31. Show that $\sqrt{8}=2\sqrt{2}$, by extracting the square roots of 8 and 2 to three decimal places.

32.* Describe a right-angled triangle whose sides are 2 inches and 3 inches. Express the length of the hypotenuse as a surd.

33. By using a right-angled triangle, how could you find a line whose length is $\sqrt{10}$ inches ?

34. If the area of a circle is 66 square inches, find the length of the radius ($\pi = 3\frac{1}{7}$).

35. The sum of the squares of two surds, one of which is double the other, is 40. Find the surds.

36. The length of the diagonal of a square is 10 inches. Find the length of the side.

37. One side of a rectangle is three times the other and the area is 96 square inches. Find the sides.

161. Like Surds. In the surd quantity $5\sqrt{3}$, 5 is a rational factor and $\sqrt{3}$ is called a surd factor.

When surds, in their simplest form, have the same surd factor, they are called **like surds** or similar surds, otherwise they are **unlike surds.**

Thus, $3\sqrt{2}$, $5\sqrt{2}$, $a\sqrt{2}$ are like surds.
and $2\sqrt{3}$, $3\sqrt{2}$, $\frac{1}{2}\sqrt{5}$ are unlike surds.

162. Addition and Subtraction of Like Surds. Like surds may be added or subtracted, the result being expressed in the form of a surd.

Thus,

$$3\sqrt{2}+5\sqrt{2}=8\sqrt{2}, \text{ just as } 3a+5a=8a.$$
$$7\sqrt{3}-4\sqrt{3}=3\sqrt{3}, \text{ just as } 7a-4a=3a.$$
$$\sqrt{75}-2\sqrt{3}=5\sqrt{3}-2\sqrt{3}=3\sqrt{3}.$$
$$\sqrt{50}+\sqrt{32}-\sqrt{18}=5\sqrt{2}+4\sqrt{2}-3\sqrt{2}=6\sqrt{2}.$$

The sum or difference of unlike surds can only be indicated.

Thus, $\sqrt{2}+\sqrt{3}$ can not be combined into a single surd, but the approximate values of $\sqrt{2}$ and $\sqrt{3}$ may be found and added.

Show that $\sqrt{2}+\sqrt{3}=\sqrt{5}$ is *not* true, by finding the square roots of 2, 3 and 5 each to two decimals.

Is it true that $\sqrt{4}+\sqrt{9}=\sqrt{13}$?

EXERCISE 111 (1-8, Oral)

Express as a single surd:

1. $3\sqrt{2}+5\sqrt{2}$.
2. $5\sqrt{7}-3\sqrt{7}$.
3. $2\sqrt{a}+3\sqrt{a}$.
4. $2\sqrt{x}+5\sqrt{x}-\sqrt{x}$.
5. $\sqrt{8}+\sqrt{2}$.
6. $\sqrt{12}+\sqrt{3}$.

7. $\sqrt{18}-\sqrt{8}$.
8. $\sqrt{4a}+\sqrt{9a}$.
9.* $\sqrt{75}+\sqrt{12}+3\sqrt{3}$.
10. $2\sqrt{18}+3\sqrt{8}-5\sqrt{2}$.
11. $\sqrt{45}-\sqrt{20}+\sqrt{80}$.
12. $2\sqrt{63}-5\sqrt{28}+\sqrt{7}$.
13. $4\sqrt{128}+4\sqrt{50}-5\sqrt{162}$.
14. $10\sqrt{44}-4\sqrt{99}$.
15. $\sqrt{45}+\sqrt{20}-\sqrt{80}+\sqrt{180}$.
16. $\sqrt{72}+\sqrt{98}-\sqrt{128}-\sqrt{32}-\sqrt{50}$.

Simplify the following and find their numerical values, correct to two decimal places, using the square root table:

17. $\sqrt{75}$.
18. $\sqrt{63}$.
19. $\sqrt{60}+\sqrt{15}$.
20. $\sqrt{147}-2\sqrt{12}$.
21. $\sqrt{128}-\sqrt{162}$.
22. $\sqrt{56}+\sqrt{72}+\sqrt{90}$.

Solve, finding x to three decimal places:

23. $x^2=37$.
24. $3x^2+5=50$.
25. $\frac{1}{2}x^2-4=19$.
26. $3\frac{1}{2}x^2=132$.
27. $\frac{1}{2}(3x^2-11)=53$.
28. $\frac{1}{4}x^2=\frac{1}{3}x^2-47$.
29. The area of a circle is 176 square inches. Find its radius.

SQUARE ROOTS OF NUMBERS FROM 1 TO 50.

n	$\sqrt{n}$	n	$\sqrt{n}$	n	$\sqrt{n}$	n	$\sqrt{n}$	n	$\sqrt{n}$
1	1·0000	11	3·3166	21	4·5826	31	5·5678	41	6·4031
2	1·4142	12	3·4641	22	4·6904	32	5·6569	42	6·4807
3	1·7321	13	3·6056	23	4·7958	33	5·7446	43	6·5574
4	2·0000	14	3·7417	24	4·8990	34	5·8310	44	6·6332
5	2·2361	15	3·8730	25	5·0000	35	5·9161	45	6·7082
6	2·4495	16	4·0000	26	5·0990	36	6·0000	46	6·7823
7	2·6458	17	4·1231	27	5·1962	37	6·0828	47	6·8557
8	2·8284	18	4·2426	28	5·2915	38	6·1644	48	6·9282
9	3·0000	19	4·3589	29	5·3852	39	6·2450	49	7·0000
10	3·1623	20	4·4721	30	5·4772	40	6·3246	50	7·0711

163. Multiplication of Surds.

$$\begin{aligned}3\sqrt{2}\times4\sqrt{3}&=3\times\sqrt{2}\times4\times\sqrt{3},\\&=3\times4\times\sqrt{2}\times\sqrt{3},\\&=12\sqrt{6}.\end{aligned}$$

It is thus seen that *the product of two surds is found by multiplying the product of the rational factors by the product of the surd factors.*

$$5\sqrt{3}\times 2\sqrt{3}=10\,.\,3=30,$$

also

$$a\sqrt{c}\times b\sqrt{c}=abc.$$

It, therefore, follows that *the product of two like surds is always a rational quantity.*

Ex. 1.—Multiply $\sqrt{50}$ by $\sqrt{75}$.

Here the surds should be simplified before multiplying.

Since $\sqrt{50}=5\sqrt{2}$ and $\sqrt{75}=5\sqrt{3}$,

$$\therefore\ \sqrt{50}\times\sqrt{75}=5\sqrt{2}\times 5\sqrt{3}=25\sqrt{6}.$$

Ex. 2.—Multiply

$2+2\sqrt{3}$ by $3-\sqrt{2}$.

Here the multiplication is performed in a manner similar to the multiplication of $a+b$ by $x+y$.

$$\begin{array}{l} 2+2\sqrt{3} \\ 3-\ \ \sqrt{2} \\ \hline 6+6\sqrt{3} \\ \qquad\quad -2\sqrt{2}-2\sqrt{6} \\ \hline 6+6\sqrt{3}-2\sqrt{2}-2\sqrt{6}. \end{array}$$

164. Conjugate Surds. If we wish to multiply

$$5\sqrt{3}+2\sqrt{2} \text{ by } 5\sqrt{3}-2\sqrt{2},$$

we may follow the same method as in the preceding example. These expressions, however, are seen to be of the same form as $a+b$ and $a-b$,

$$\therefore\ (5\sqrt{3}+2\sqrt{2})(5\sqrt{3}-2\sqrt{2})=(5\sqrt{3})^2-(2\sqrt{2})^2=75-8=67.$$

Similarly, $(3+\sqrt{2})(3-\sqrt{2})=9-2=7,$

and $(2-\sqrt{10})(2+\sqrt{10})=4-10=-6.$

Such surd quantities as these which differ only in the sign which connects their terms are called **conjugate surds.**

Note that *the product of two conjugate surds is always a rational quantity.*

EXERCISE 112 (1-12, Oral)

Find the product of:

1. $2\sqrt{3}, 3\sqrt{5}$.
2. $5\sqrt{2}, 6\sqrt{3}$.
3. $a\sqrt{b}, b\sqrt{a}$.
4. $3\sqrt{2}, \sqrt{3}, \sqrt{5}$.
5. $(2\sqrt{3})^2, (\sqrt{2})^2$.
6. $\sqrt{2}+1, \sqrt{2}$.
7. $\sqrt{3}+\sqrt{5}, \sqrt{2}$.
8. $\sqrt{a}+\sqrt{b}-1, \sqrt{c}$.
9. $\sqrt{3}+\sqrt{2}, \sqrt{3}-\sqrt{2}$.
10. $\sqrt{10}-3, \sqrt{10}+3$.
11. $\sqrt{x}-\sqrt{y}, \sqrt{x}+\sqrt{y}$.
12. $2\sqrt{2}+\sqrt{3}, 2\sqrt{2}-\sqrt{3}$.
13.* $3\sqrt{6}, 4\sqrt{2}$.
14. $3\sqrt{2}, 4\sqrt{7}, \frac{1}{2}\sqrt{2}$.
15. $(\sqrt{3}+\sqrt{2})^2$.
16. $(2\sqrt{5}-\sqrt{7})^2$.
17. $(3\sqrt{2}+2\sqrt{3})^2$.
18. $(\sqrt{a}+\sqrt{b})^2$.
19. $4+3\sqrt{2}, 5-3\sqrt{2}$.
20. $3\sqrt{2}+2\sqrt{3}, 5\sqrt{2}-3\sqrt{3}$.
21. $3\sqrt{5}-4\sqrt{2}, 2\sqrt{5}+3\sqrt{2}$.
22. $3\sqrt{a}-2\sqrt{b}, 2\sqrt{a}-3\sqrt{b}$.
23. $\sqrt{5}+\sqrt{3}+\sqrt{2}, \sqrt{5}+\sqrt{3}-\sqrt{2}$.
24. $\sqrt{7}+2\sqrt{2}-\sqrt{3}, \sqrt{7}-2\sqrt{2}+\sqrt{3}$.
25. $\sqrt{a+b}-3, \sqrt{a+b}+2$.
26. $\sqrt{a}+\sqrt{a-1}, \sqrt{a}-\sqrt{a-1}$.
27. $(\sqrt{3}+\sqrt{2}+1)^2$.
28. $(\sqrt{5}+2\sqrt{2}-\sqrt{3})^2$.
29. $(\sqrt{a+b}+\sqrt{a-b})^2$.
30. $(3\sqrt{x-y}-2\sqrt{x+y})^2$.

Simplify:

31. $(6-2\sqrt{3})(6+2\sqrt{3})-(5-\sqrt{2})(5+\sqrt{2})$.

32. $(\sqrt{3}-\sqrt{2}+1)^2+(\sqrt{3}+\sqrt{2}-1)^2$.

33. $(\sqrt{50}-\sqrt{18}+\sqrt{72}+\sqrt{32})\times\frac{1}{2}\sqrt{3}$.

34. $2(4\sqrt{3}+3\sqrt{2})(3\sqrt{3}-2\sqrt{2})+(5\sqrt{2}-3\sqrt{3})(4\sqrt{2}+2\sqrt{3})$.

35. $(\sqrt{3}+\sqrt{2})(2\sqrt{3}-\sqrt{2})(\sqrt{3}-2\sqrt{2})(\sqrt{3}-3\sqrt{2})$.

36. By squaring $\sqrt{10}+\sqrt{5}$ and $\sqrt{8}+\sqrt{7}$, find which is the greater.

37. The product of $5\sqrt{3}+3\sqrt{7}$ and $3\sqrt{3}-\sqrt{7}$ lies between what two consecutive integers?

38. Find the area of a rectangle whose sides are $5+\sqrt{2}$ and $10-2\sqrt{2}$ inches

39. The sides of a right-angled triangle are $7+4\sqrt{2}$ and $7-4\sqrt{2}$ inches. Find the hypotenuse.

40. The base of a triangle is $2\sqrt{3}+3\sqrt{2}$ inches and the altitude is $3\sqrt{3}+2\sqrt{2}$ inches. Find the area to two decimal places.

165. Division of Surds.

Since $\sqrt{a}\times\sqrt{b}=\sqrt{ab}$, $\therefore\ \sqrt{ab}\div\sqrt{a}=\sqrt{\dfrac{ab}{a}}=\sqrt{b}$.

Similarly, $$\sqrt{a}\div\sqrt{b}=\frac{\sqrt{a}}{\sqrt{b}}=\sqrt{\frac{a}{b}},$$

and $$3\sqrt{15}\div2\sqrt{5}=\tfrac{3}{2}\sqrt{3}.$$

Ex. 1.—Find the numerical value of $\sqrt{5}\div\sqrt{2}$ or $\dfrac{\sqrt{5}}{\sqrt{2}}$.

(1) We might find the square roots of 5 and 2 and perform the required division.

$$\sqrt{5}\div\sqrt{2}=2{\cdot}236\div1{\cdot}414=1{\cdot}581$$

(2) $$\sqrt{5}\div\sqrt{2}=\sqrt{\tfrac{5}{2}}=\sqrt{2{\cdot}5}=1{\cdot}581.$$

(3) $$\frac{\sqrt{5}}{\sqrt{2}}=\frac{\sqrt{5}\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}}=\frac{\sqrt{10}}{2}=\frac{3{\cdot}162}{2}=1{\cdot}581.$$

Here the third method is at once seen to be simpler than either of the others.

In (3) we changed $\dfrac{\sqrt{5}}{\sqrt{2}}$ into $\dfrac{\sqrt{10}}{2}$, that is, we made the denominator a rational quantity. This operation is called **rationalizing the denominator.**

Ex. 2.—Find the value of $\dfrac{1}{\sqrt{2}}$, if $\sqrt{2}=1{\cdot}4142$.

Here, instead of dividing 1 by 1·4142, we first rationalize the denominator.

Then $$\frac{1}{\sqrt{2}}=\frac{1\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}}=\frac{\sqrt{2}}{2}=\frac{1{\cdot}4142}{2}=\cdot7071.$$

Ex. 3.—Divide $6\sqrt{8}$ by $10\sqrt{27}$.

$$\frac{6\sqrt{8}}{10\sqrt{27}}=\frac{6\times2\sqrt{2}}{10\times3\sqrt{3}}=\frac{2\sqrt{2}}{5\sqrt{3}}=\frac{2\times\sqrt{2}\times\sqrt{3}}{5\times\sqrt{3}\times\sqrt{3}}=\frac{2\sqrt{6}}{15}=\frac{2\times2{\cdot}4495}{15}=\cdot3266$$

Ex. 4.—Rationalize the denominator of $\frac{2+\sqrt{5}}{3+\sqrt{5}}$.

We have already seen that the denominator will be rational if we multiply it by its conjugate $3-\sqrt{5}$.

$$\therefore \quad \frac{2+\sqrt{5}}{3+\sqrt{5}}=\frac{(2+\sqrt{5})(3-\sqrt{5})}{(3+\sqrt{5})(3-\sqrt{5})}=\frac{1+\sqrt{5}}{9-5}=\frac{1+\sqrt{5}}{4}.$$

Ex. 5.—Divide $5+2\sqrt{3}$ by $7-4\sqrt{3}$.

Write the quotient in the fractional form $\frac{5+2\sqrt{3}}{7-4\sqrt{3}}$, rationalize the denominator and simplify.

EXERCISE 118 (1-12, Oral)

Divide :

1. $3\sqrt{27}\div\sqrt{3}$.
2. $\sqrt{12}\div\sqrt{3}$.
3. $\sqrt{72}\div3\sqrt{8}$.
4. $\sqrt{abc}\div\sqrt{a}$.
5. $\sqrt{18}+\sqrt{12}$ by $\sqrt{3}$.
6. $\sqrt{ab}+\sqrt{ac}$ by $\sqrt{a}$.

Rationalize the denominator of :

7. $\frac{2}{\sqrt{3}}$.
8. $\frac{10}{\sqrt{5}}$.
9. $\frac{\sqrt{5}}{\sqrt{3}}$.
10. $\frac{a}{\sqrt{b}}$.
11. $\frac{3\sqrt{5}}{\sqrt{6}}$.
12. $\frac{1}{\sqrt{2}-1}$.

13.* $\frac{2}{7-4\sqrt{3}}$.

14. $\frac{12}{3\sqrt{2}-2\sqrt{3}}$.
15. $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.
16. $\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}$.
17. $\frac{5\sqrt{3}-3\sqrt{5}}{\sqrt{5}-\sqrt{3}}$.
18. $\frac{\sqrt{7}+\sqrt{2}}{9+2\sqrt{14}}$.

Find the value to three decimal places, using the table :

19. $\frac{1}{\sqrt{3}}$.
20. $\frac{15}{\sqrt{18}}$.
21. $\frac{2\sqrt{3}}{3\sqrt{2}}$.
22. $\frac{1}{\sqrt{3}+\sqrt{2}}$.
23. $\frac{17}{3\sqrt{7}+2\sqrt{3}}$.
24. $\frac{\sqrt{7}-\sqrt{5}}{\sqrt{7}+\sqrt{5}}$.
25. $\sqrt{3}\div\sqrt{2}$.
26. $2\sqrt{63}\div3\sqrt{35}$.
27. $1\div(7+4\sqrt{3})$.

Solve, giving the value of x to two decimal places, using the table:

28. $x\sqrt{2}=3$. **29.** $x\sqrt{3}=\sqrt{2}$. **30.** $x\sqrt{3}=\sqrt{2}+1$.

31. $x\sqrt{3}-x\sqrt{2}=1$. **32.** $x\sqrt{5}-5=2x-\sqrt{5}$.

33. $x^2(\sqrt{3}-1)=2(\sqrt{3}+1)$.

34. The area of a triangle is 2 square feet. The altitude is $\sqrt{5}+\sqrt{3}$ feet. Find the base to three decimals.

35. Simplify
$$\frac{2+\sqrt{10}}{4\sqrt{2}+\sqrt{20}-\sqrt{18}-\sqrt{5}}.$$

166. Surd Equations. A surd equation is one in which the unknown quantity is found under the root sign, in one or more of the terms.

Thus, $\sqrt{x+7}=4$, $\sqrt{x}+\sqrt{x-5}=5$, are surd equations.

Ex. 1.—Solve $\sqrt{x-3}=2$.

Square both sides, $x-3=4$,

$\therefore\ x=7$.

Verification: $\sqrt{x-3}=\sqrt{7-3}=\sqrt{4}=2$.

Ex. 2.—Solve $\sqrt{5x-1}-2\sqrt{x+3}=0$.

Transpose $2\sqrt{x+3}$, $\sqrt{5x-1}=2\sqrt{x+3}$.

Squaring, $5x-1=4x+12$,

$\therefore\ x=13$.

Verification: $\sqrt{5x-1}-2\sqrt{x+3}=\sqrt{64}-2\sqrt{16}=8-8=0$.

Note that in verifying we have taken the positive square root only, as defined in art. 63.

EXERCISE 114 (1-8, Oral)

Solve and verify:

1. $2\sqrt{x}=6$. **2.** $\sqrt{x}-5=4$. **3.** $6-\sqrt{x}=1$.

4. $\sqrt{x}+2=4$. **5.** $4\sqrt{x}=\sqrt{20}$. **6.** $\sqrt{x}-b=a$.

7. $m+\sqrt{x}=n$. **8.** $7-\sqrt{x-4}=3$.

9. $\sqrt{x^2+9}=9-x$. **10.** $\sqrt{x^2+11x+3}=x+5$.

11. $\sqrt{9x^2-11x-5}=3x-2$. **12.** $2x-\sqrt{4x^2-10x+4}=4$.

13. $2a+\sqrt{x+a^2}=b+a$. **14.** $\sqrt{(x-a)^2+2ab+b^2}=x-a+b$.

EXERCISE 115 (Review of Chapter XVIII)

Simplify :

1.* $\sqrt{8}+\sqrt{18}+\sqrt{98}$.

2. $\sqrt{500}+\sqrt{80}-\sqrt{20}$.

3. $5\sqrt{3}+3\sqrt{27}-\sqrt{48}$.

4. $(4\sqrt{5}+\sqrt{18})(4\sqrt{5}-\sqrt{18})$.

5. $(6\sqrt{6}-5)(6\sqrt{6}+5)$.

6. $(\sqrt{6}+\sqrt{2}+2)(\sqrt{6}-\sqrt{2}-2)$.

7. $(\sqrt{8}+\sqrt{2}-2)^2$.

8. $(\sqrt{3}-2\sqrt{2}-1)^2$.

9. $5\sqrt{27}\div 6\sqrt{75}$.

10. $(\sqrt{5}-2)\div(\sqrt{5}+2)$.

11. $(\sqrt{125}+\sqrt{45})\div\sqrt{320}$.

12. $(5+\sqrt{3})(5-\sqrt{3})\div(\sqrt{13}-\sqrt{2})$

13. Multiply $3\sqrt{8}+2\sqrt{3}-\sqrt{2}$ by $2\sqrt{8}-\sqrt{3}+4\sqrt{2}$.

14. By how much does the square of $\sqrt{3}+\dfrac{1}{\sqrt{3}}$ exceed the square of $\sqrt{2}+\dfrac{1}{\sqrt{2}}$?

15. Show by multiplication that the value of $\sqrt{3}$ lies between 1·732 and 1·733. Which of these is the closer approximation to $\sqrt{3}$?

16. Which is the greater, $\sqrt{14}+\sqrt{8}$ or $\sqrt{12}+\sqrt{10}$?

17. The product of $3\sqrt{2}-2\sqrt{3}$ and $2\sqrt{3}-\sqrt{2}$ lies between what two consecutive integers ?

18. Rationalize the denominators of :

$$\frac{4}{\sqrt{2}},\ \frac{3\sqrt{2}}{2\sqrt{3}},\ \frac{3}{2}\sqrt{\frac{5}{6}},\ \sqrt{3\cdot 5},\ \frac{2\sqrt{6}-2}{3\sqrt{3}+\sqrt{2}}.$$

Solve and verify :

19. $\sqrt{x+3}=4$.

20. $\sqrt{3x-2}=2\sqrt{x-2}$.

21. $\sqrt{x^2-5}+1=x$.

22. $\sqrt{2x+7}=3\sqrt{x}$.

23. $\sqrt{x^2-5x+11}=x+2$.

24. $\sqrt{x^2-2}=1-x$.

25. Using the table, solve : $x^2=75$, $x^2=63$, $\frac{1}{2}x^2=49$, $x\sqrt{3}=\sqrt{5}$, $x\sqrt{2}+1=\sqrt{3}$.

26. Find to three decimal places the values of :

$$\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{5}},\ \frac{1}{\sqrt{2}+1},\ \frac{2\sqrt{10}-\sqrt{5}}{\sqrt{10}+\sqrt{5}},\ \frac{3\sqrt{2}-2}{4\sqrt{2}+1}.$$

27. Find the value of

$$(2\sqrt{2}+\sqrt{3})(3\sqrt{2}-\sqrt{3})(3\sqrt{3}-\sqrt{2}).$$

28. If the sides of a right-angled triangle are $\sqrt{3}+1$ and $\sqrt{3}-1$, what is the length of the hypotenuse?

29. Simplify $\dfrac{\sqrt{5}-1}{\sqrt{5}-2}-\dfrac{\sqrt{5}-3}{\sqrt{5}+3}$ and $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}-\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$.

30. Find the value to two decimal places of

$$\frac{x+y}{x-y}+\frac{x-y}{x+y}, \text{ when } x=2+\sqrt{3},\ y=2-\sqrt{3}.$$

31. Multiply $2\sqrt{30}-3\sqrt{5}+5\sqrt{3}$ by $\sqrt{3}+2\sqrt{2}-\sqrt{5}$.

32. Multiply $\sqrt{7+2\sqrt{6}}$ by $\sqrt{7-2\sqrt{6}}$.

33. The area of a rectangle is $16\sqrt{10}-25$ and one side is $3\sqrt{5}-\sqrt{2}$. Find the other side to two decimal places.

CHAPTER XIX

QUADRATIC EQUATIONS

167. A quadratic equation has already been defined in art. 104. In the same article we considered the method of solving some of the simpler forms of it.

Quadratic equations frequently occur in the solution of problems as shown in the following examples.

Ex. 1.—Find two consecutive numbers whose product is 462.

Let the numbers be x and $x+1$.

$$\therefore \quad x(x+1)=462,$$
$$\therefore \quad x^2+x-462=0.$$

Ex. 2.—The length of a rectangle is 10 feet more than the width and the area is 875 square feet. Find the dimensions.

Let $x=$ the number of feet in the width,

$\therefore \quad x+10=$ the number of feet in the length,

$$\therefore \quad x(x+10)=875,$$
$$\therefore \quad x^2+10x-875=0.$$

Ex. 3.—Divide 20 into two parts so that the sum of their squares may be 36 more than twice their product.

Let $x=$ one part,

$\therefore \quad 20-x=$ the other part,

$$\therefore \quad x^2+(20-x)^2=2x(20-x)+36,$$
$$\therefore \quad x^2+400-40x+x^2=40x-2x^2+36,$$
$$\therefore \quad 4x^2-80x+364=0,$$
$$\therefore \quad x^2-20x+91=0.$$

EXERCISE 116

Represent the number to be found by x and obtain, in its simplest form, the quadratic equation which must be solved in each of the following:

1.* The sum of a number and its square is 132. Find the number.

2. Find the number which is 156 less than its square.

3. The sum of the squares of three consecutive numbers is 149. Find the middle number.

4. The product of a number and the number increased by 6 is 112. Find the number.

5. The length of a rectangle is 6 feet less than five times the width. The area is 440 square feet. Find the width.

6. The average number of words on each page of a book is 6 more than the number of pages. The total number of words is 9400. Find the number of pages.

7. The area of a rectangle is 88 square inches and the perimeter is 38 inches. Find the length.

168. Standard Form of the Quadratic Equation. Every quadratic equation may be reduced to the form

$$ax^2+bx+c=0,$$

in which a, b and c are any known numbers, except that a can not be zero.

The term not containing x is called the **absolute term.**

It is frequently necessary to simplify equations to bring them to the standard form, and thus determine if they are quadratic equations.

Ex. 1.

$$(x+1)(2x+3)=4x^2-22,$$

$$\therefore \quad 2x^2+5x+3=4x^2-22,$$

$$\therefore \quad -2x^2+5x+25=0,$$

$$\therefore \quad 2x^2-5x-25=0.$$

Here the equation is seen to be a quadratic. The coefficient of x^2 is 2, of x is -5 and the absolute term is -25.

Or, $a=2$, $b=-5$, $c=-25$.

Ex. 2.

$$\frac{7x}{4}+\frac{x-7}{x}=1,$$

$$\therefore\ 7x^2+4(x-7)=4x,$$

$$\therefore\ 7x^2-28=0,$$

$$\therefore\ x^2-4=0.$$

Here $a=1$, $b=0$, $c=-4$.

Ex. 3.

$$\frac{2x}{x-1}+\frac{x+1}{x+2}=3,$$

$$\therefore\ 2x(x+2)+(x+1)(x-1)=3(x-1)(x+2),$$

$$\therefore\ 2x^2+4x+x^2-1=3x^2+3x-6,$$

$$\therefore\ x+5=0.$$

Here $a=0$, $b=1$, $c=5$, and the equation is not a quadratic, since the coefficient of x^2 is zero.

EXERCISE 117

Reduce to the standard form and state the values of a, b and c, in which a is always positive:

1.* $6x^2=x+22$.

2. $25x=6x^2+21$.

3. $19x=15-8x^2$.

4. $2=11x-12x^2$.

5. $\dfrac{x^2+9}{2}=5x$.

6. $x+\dfrac{1}{x}=\dfrac{5}{2}$.

7. $8-x=\dfrac{x^2}{4}$.

8. $(3x-5)(2x-5)=x^2+2x-3$.

9. $\dfrac{3x-8}{x-2}=\dfrac{5x-2}{x+5}$.

10. $\dfrac{4}{x-1}-\dfrac{5}{x+2}=\dfrac{3}{x}$.

11. $\dfrac{x+1}{x+2}+\dfrac{3x}{x-1}=4$.

12. $\dfrac{2x}{x-3}-\dfrac{x-3}{x}=1$.

169. Solution by Factoring.

(1) When the absolute term is zero, the equation ca always be solved by factoring.

Ex. 1.—Solve $2x^2-3x=0.$

$$x(2x-3)=0,$$
$$\therefore\ x=0 \text{ or } 2x-3=0,$$
$$\therefore\ x=0 \text{ or } \tfrac{3}{2}.$$

Verify both roots.

Ex. 2.—Solve $ax^2+bx=0.$

$$x(ax+b)=0,$$
$$\therefore\ x=0 \text{ or } ax+b=0,$$
$$\therefore\ x=0 \text{ or } -\frac{b}{a}.$$

(2) When the middle term is zero, the equation can always be solved by factoring, or by extracting the square root.

Ex.—Solve $3x^2-27=0.$

$$3(x-3)(x+3)=0,$$
$$\therefore\ x-3=0 \text{ or } x+3=0,$$
$$\therefore\ x=\pm 3.$$

Or thus,

$$3x^2-27=0,$$
$$\therefore\ x^2=9,$$
$$\therefore\ x=\pm 3.$$

(3) The equation is a complete quadratic when none of the coefficients a, b, c is zero. If the quadratic expression, ax^2+bx+c, can be factored by any of the methods previously given, the solution is then easily effected.

Ex. 1.—Solve $3x^2-11x=14.$

$$3x^2-11x-14=0,$$
$$\therefore\ (x+1)(3x-14)=0,$$
$$\therefore\ x=-1 \text{ or } \tfrac{14}{3}.$$

Verify both of these roots.

Ex. 2.—Solve $x^2-mx+nx-mn=0.$

$$x(x-m)+n(x-m)=0,$$
$$\therefore\ (x-m)(x+n)=0,$$
$$\therefore\ x=m \text{ or } -n.$$

EXERCISE 112

1-12. Solve the equations in the preceding exercise and verify.

13-19. Solve the problems in the first exercise in this Chapter. (Verify the results.)

Solve by factoring and verify:

20. $x^2-3ax+2a^2=0.$
21. $x^2-b^2=0.$
22. $x^2-mx-6m^2=0.$
23. $x^2-ax-bx+ab=0.$
24. $x^2+2x(a+b)+4ab=0.$
25. $2ax^2+ax-2x=1.$
26. $(x-a)(x-b)=ab.$
27. $x^2-a^2=(x-a)(b+c).$

170. Consider the problem: Find two numbers whose sum is 100 and whose product is 2491.

Let $x=$ one number,

$\therefore\ 100-x=$ the other number,

$\therefore\ x(100-x)=2491,$

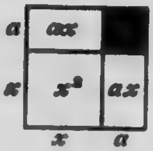
$\therefore\ x^2-100x+2491=0.$

To solve this equation by the preceding method, we must find two factors of 2491 whose sum is 100, but this is exactly what the problem requires us to find.

The necessity is therefore seen for another method of solving the quadratic equation when the factors of the quadratic expression cannot be obtained readily by inspection.

171. Solution by Completing the Square. We know that $(x+a)^2=x^2+2ax+a^2$, the middle term being *twice* the product of x and a.

If the first two terms of a square are x^2+2ax, we know that it must be the square of $x+a$, and, therefore, a^2 must be added to x^2+2ax to make a complete square.



What is the area of the shaded portion in the diagram?

Similarly, x^2+4x must be the first two terms in the square of $x+2$. To make x^2+4x a complete square we must add 2^2 or 4. Also, x^2-8x are the first two terms in the square of $x-4$, and, therefore, 4^2 or 16 must be added.

To complete the square, it is seen that the quantity to be added is the square of half of the coefficient of x.

Ex. 1.—Factor $x^2+6x-40$.

Add 9 to x^2+6x to make a complete square.

Then

$$x^2+6x-40=x^2+6x+9-9-40,$$
$$=x^2+6x+9-49,$$
$$=(x+3)^2-7^2,$$
$$=(x+3+7)(x+3-7),$$
$$=(x+10)(x-4).$$

Ex. 2.—Factor $x^2+5x-806$.

Add $(\frac{5}{2})^2$ or $\frac{25}{4}$ to x^2+5x to complete the square.

Then

$$x^2+5x-806=x^2+5x+\tfrac{25}{4}-\tfrac{25}{4}-806,$$
$$=x^2+5x+\tfrac{25}{4}-\tfrac{3249}{4},$$
$$=(x+\tfrac{5}{2})^2-(\tfrac{57}{2})^2,$$
$$=(x+\tfrac{5}{2}+\tfrac{57}{2})(x+\tfrac{5}{2}-\tfrac{57}{2}),$$
$$=(x+31)(x-26).$$

Ex. 3.—Solve $x^2-100x+2491=0$.

Add 50^2 or 2500 to complete the square.

$$x^2-100x+2500-2500+2491=0,$$
$$\therefore \quad x^2-100x+2500-9=0,$$
$$\therefore \quad (x-50)^2-3^2=0,$$
$$\therefore \quad (x-50+3)(x-50-3)=0,$$
$$\therefore \quad (x-47)(x-53)=0,$$
$$\therefore \quad x-47=0 \text{ or } x-53=0,$$
$$\therefore \quad x=47 \text{ or } 53.$$

The solution might be contracted by writing it in the following form:

$$x^2-100x+2491=0.$$

Transpose the absolute term, $\therefore\ x^2-100x=-2491.$

Add 2500 to each side, $\therefore\ x^2-100x+2500=-2491+2500=9.$

Take the square root, $\therefore\ x-50=\pm 3,$

$$\therefore \quad x=50\pm 3,$$
$$=53 \text{ or } 47.$$

Here the solution depends upon the same principle, but assumes a simpler form.

It is thus seen that *we effect the solution of a quadratic equation by finding and solving the two simple equations of which it is composed.*

Thus by the first method of solving $x^2-100x+2491=0$, we obtained the two simple equations $x-47=0$ and $x-53=0$, and by the second $x-50=3$, and $x-50=-3$.

Ex. 4.—Solve $3x^2+x=10$.

Divide by 3 to make the first term a square,

$\therefore \quad x^2+\frac{1}{3}x=\frac{10}{3}.$

Add $(\frac{1}{6})^2$ to each side, $\therefore \quad x^2+\frac{1}{3}x+\frac{1}{36}=\frac{10}{3}+\frac{1}{36}=\frac{121}{36}.$

Take the square root, $\therefore \quad x+\frac{1}{6}=\pm\frac{11}{6},$

$\therefore \quad x=\pm\frac{11}{6}-\frac{1}{6}=\frac{5}{3}$ or -2.

Verify both of these roots.

The steps in this method are :

1. *Reduce the equation to the standard form and remove the absolute term to the right.*

2. *Divide by the coefficient of* x^2 *if not unity.*

3. *Complete the square by adding to each side the square of half the coefficient of* x.

4. *Take the square root of each side.*

5. *Solve the resulting simple equations.*

EXERCISE 119 (1-8, Oral)

What must be added to each of the following to make a complete square ?

1. x^2+2x.
2. x^2-4x.
3. x^2+10x.
4. x^2-14x.
5 x^2+2x.
6. x^2-5x.
7. x^2+4ax.
8. $x^2-\frac{1}{2}x$

Factor, by making the difference of squares, and verify :

9. $x^2+4x-77$.
10. $x^2-54x+713$.
11. $x^2-2x-899$
12. $x^2-x-1640$.
13. $x^2-\frac{11}{3}x+\frac{10}{3}$.
14. $3x^2+16x-99$.

Use the method of completing the square to solve the following and verify the roots:

15. $x^2+8x=9$.
16. $x^2-6x=7$.
17. $x^2-10x+9=0$.
18. $x^2-9x+18=0$.
19. $x^2+7x+10=0$.
20. $x^2-x=2$.
21. $2x^2-3x=2$.
22. $2x^2+x=1081$.
23. $6x^2+5x=6$.
24. If $x^2+x=1\frac{1}{3}$, find the values of $x+\frac{1}{x}$.

172. Equations with Irrational Roots. In all the quadratic equations we have solved, we found that when we had completed the square on the left side, the quantity on the right was also a square. This would not always be the case.

Ex. 1.—Solve $x^2-6x-1=0$.

$$x^2-6x=1,$$
$$\therefore\quad x^2-6x+9=10,$$
$$\therefore\quad x-3=\pm\sqrt{10},$$
$$\therefore\quad x=3\pm\sqrt{10}.$$

The two roots are $3+\sqrt{10}$ and $3-\sqrt{10}$.

We might go a step further and substitute for $\sqrt{10}$ its approximate value 3·16.

The two roots would then be

$$3\pm 3{\cdot}16=6{\cdot}16 \text{ or } -{\cdot}16.$$

If we substitute either of these values for x in x^2-6x-1, the result will not be exactly 0, as we might expect, because $\sqrt{10}$ is not exactly 3·16, but the difference between 0 and the value found for x^2-6x-1 will be very small.

Ex. 2.—Solve $2x^2+x=2$.

$$x^2+\tfrac{1}{2}x=1,$$
$$\therefore\quad x^2+\tfrac{1}{2}x+\tfrac{1}{16}=\tfrac{17}{16},$$
$$\therefore\quad x+\tfrac{1}{4}=\pm\sqrt{\tfrac{17}{16}}=\pm\tfrac{1}{4}\sqrt{17}.$$
$$\therefore\quad x=-\tfrac{1}{4}\pm\tfrac{1}{4}\sqrt{17}.$$

The two roots are $-\frac{1}{4}+\frac{1}{4}\sqrt{17}$, $-\frac{1}{4}-\frac{1}{4}\sqrt{17}$, or ·781, $-1{\cdot}281$, on substituting $\sqrt{17}=4{\cdot}123$.

173. Inadmissible Solutions of Problems. When a problem is solved by means of a quadratic equation, it does not follow that the two roots of the equation will furnish two admissible solutions of the problem.

Ex.—A man walked 25 miles. If his rate had been one mile per hour faster he would have completed the journey in $1\frac{1}{4}$ hours less. What was his rate ?

Let his rate be x miles per hour.

The time taken to walk 25 miles $= \frac{25}{x}$ hr.

At the supposed rate his time $= \frac{25}{x+1}$ hr.

$$\therefore \quad \frac{25}{x} - \frac{25}{x+1} = 1\tfrac{1}{4}.$$

Simplifying, $\quad x^2+x-20=0,$

Solving, $\quad x=4 \text{ or } -5.$

Therefore his rate was 4 miles per hour, the other root giving a solution which is inadmissible

EXERCISE 120

Solve, finding the roots approximately to three decimal places, using the table :

1.* $x^2-4x=1.$ 2. $x^2-10x+17=0.$ 3. $x^2+2x-6=0.$

4. $x^2+8x=19.$ 5. $x(x+3)=\frac{3}{4}.$ 6. $2x^2+3x-4=0.$

Solve, expressing the roots in the surd form :

7. $x^2-6x=2.$ 8. $x^2+8x=11.$ 9. $4x^2-4x=7.$

10. $4x^2-8x=37.$ 11. $3x^2-5x-11=0.$ 12. $\frac{1}{2}x^2+\frac{1}{3}x=\frac{1}{4}.$

The following problems reduce to quadratic equations. In solving the equations factor by inspection where possible and verify the results.

13. The sum of two numbers is 11 and their product is 30. Find the numbers.

14. The sum of the squares of two consecutive numbers is 85. Find the numbers.

15. The difference between the sides of a rectangle is 13 inches and the area is 300 square inches. Find the sides.

16. Find two consecutive numbers such that the square of their sum exceeds the sum of their squares by 220.

17. A merchant bought silk for $54. The number of cents in the price per yard exceeded the number of yards by 30. Find the number of yards.

18. The area of a rectangular field is 9 acres and the length is 18 rods more than the width. Find the length.

19. The three sides of a right-angled triangle are consecutive integers. Find the sides.

20. How can you form 730 men into two solid squares so that the front of one will contain 4 men more than the front of the other ?

21. The owner of a rectangular lot 12 rods by 5 rods wishes to double the size of the lot by increasing the length and width by the same amount. What should the increase be ?

22. If $x+2$ men in $x+5$ days do five times as much work as $x+1$ men in $x-1$ days, find x.

23. A rectangular mirror 18 inches by 12 inches is to be surrounded by a frame of uniform width whose area is equal to that of the mirror. Find the width of the frame.

24. What must be the radius of a circle in order that a circle with a radius 3 inches less may be $\frac{4}{9}$ as large ?

25. One side of a right triangle is 10 less than the hypotenuse and the other is 5 less. Find the sides.

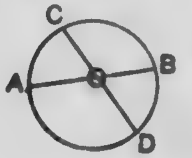
26. A man spends $90 for coal, and finds that when the price is increased $1·50 per ton he will get 3 tons less for the same money. What was the price per ton ?

27. A man bought a number of articles for $200. He kept 5 and sold the remainder for $180, gaining $2 on each. How many did he buy ?

28. The sum of the two digits of a number is 9. The sum of the squares of the digits is $\frac{5}{7}$ of the number. Find the number.

29. A number of cattle cost $400, but 2 having died the rest averaged $10 per head more. Find the number bought.

30. How much must be added to the length of a rectangle 8 inches by 6 inches in order to increase the diagonal by 2 inches ?

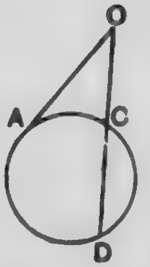


31. In the figure, the rectangle $AO \, . \, OB$ = rectangle $CO \, . \, OD$.

(1) If $AO=16$, $BO=3$, $CO=8$, find OD.

(2) If $AO=10$, $BO=4$, $CD=13$, find OD.

32. In the figure, when OA is a tangent to the circle, $OA^2=OC \, . \, OD$.



(1) If $OC=4$, $CD=5$, find OA.

(2) If $OA=8$, $OD=10$, find OC.

(3) If $OA=15$, $CD=16$, find OD.

33. I sold an article for $56 and gained a per cent. equal to the cost in dollars. What was the cost ?

34. The denominator of a fraction exceeds the numerator by 3. If 4 is added to each term the resulting fraction is $\frac{6}{5}$ of the original fraction. Find the fraction.

35. An open box containing 432 cubic inches is to be made from a square piece of tin by cutting out a 3 inch square from each corner and turning up the sides. How large a piece of tin must be used ?

36. A and B can together do a piece of work in $14\frac{2}{5}$ days, and A alone can do it in 12 days less than B. Find the time in which A could do it alone.

EXERCISE 121 (Review of Chapter XIX)

1. What is a quadratic equation ?

2. Is $(x+1)(x-2)(x+3)=(x-4)(x-1)(x+7)$ a quadratic equation? Solve it.

3. The sum of a positive number and its square is 4. Find the number to two decimal places.

4. Solve $\frac{5}{5-x}+\frac{8}{8-x}=3$; $\frac{1}{x-9}-\frac{1}{x-8}=\frac{1}{12}$.

5. If $x^2y^2-6xy-7=0$, what are the values of xy ?

6. Are $x=4$ and $x^2=16$ equivalent equations, that is, have they the same roots ?

7. Solve $x^2-xy+y^2=39$, when $y=7$.

8. Divide 14 into two parts so that the sum of their squares may be greater than twice their product by 4.

9. If $(x-2)(x-3)=7(x-3)$, does it follow that $x-2=7$? What is the proper conclusion ?

10. The distance (s) in feet that a body falls from rest in t seconds is given by the formula $s=16{\cdot}1t^2$. How long will it take a body to fall 6440 feet ?

11. Solve $3x^2-4x-1=0$.

12. Ten times a number is 24 greater than the square of the number. Does this condition determine the number definitely ?

13. Solve $\dfrac{x+2}{x-1}=\dfrac{7}{3}+\dfrac{4-x}{2x}$.

14. Find two consecutive odd numbers whose product is 399.

15. Solve $(2x+3)^2-2(2x+3)=35$.

16. The units digit of a number is the square of the tens digit and the sum of the digits is 12. Find the number.

17. Solve $\dfrac{x+10}{x-5}-\dfrac{10}{x}=\dfrac{11}{6}$; $\dfrac{7}{x+5}-\dfrac{1}{x-3}=\dfrac{5}{3}$.

18. If a train travelled 10 miles per hour faster it would require 2 hours less to travel 315 miles. Find the rate.

19. Solve $(3x-7)(2x-9)-(5x-12)(x-6)=(x-2)(2x-3)$.

20.* Find, to three decimal places, the positive number which is less than its square by unity.

21. If $4x^2-3xy+y^2=14$, find x if $y=x+3$.

22. The perimeter of a rectangle is 34 feet and the length of the diagonal is 13 feet. Find the sides.

23. Solve $x^2+(x-4)^2=40$. State the problem, the condition in which is expressed by this equation.

24. A line 20 inches long is divided into two parts, such that the rectangle contained by the parts has an area of 48 square inches more than the square on the shorter part. Find the lengths of the parts.

25. Solve $x^3+y^3=9$, when $y=3-x$.

26. The diagonal of a rectangle is 39 feet and the shorter side is $\frac{5}{12}$ of the longer. Find the area.

27. If 5 is one root of $x^3-7x^2+6x+20=0$, find the other roots to three decimal places.

28. Find the price of eggs per dozen when 10 less in a dollar's worth raises the price 4 cents per dozen.

29. The length of a field exceeds its breadth by 30 yards. If the field were square but of the same perimeter, its area would be $\frac{1}{15}$ greater. Find the sides.

30. If $8x-\frac{11}{x}=4$, find x to three decimal places.

31. The cost of an entertainment was $20. This was to be divided equally among the men present. But four failed to contribute anything, and thereby the cost to each of the others was increased 25 cents. How many men were there ?

32. If a man walked one mile per hour faster he would walk 36 miles in 3 hours less time. What is his rate of walking ?

33. A polygon with n sides has $\frac{1}{2}n(n-3)$ diagonals. If a polygon has 20 diagonals, how many sides has it ?

34. Solve $a^2(x-a)^2=b^2(x+a)^2$.

35. *A* can do a piece of work in 10 days less than *B*. If they work together they can do it in 12 days. In what time could each do it alone ?

36. If $x^3+\frac{1}{x^3}=8\frac{1}{8}$, find the value of x^3 and of x.

37. The length of a rectangular field is to the width as 3 to 2 and the area is 5·4 acres. How many rods longer must it be to contain 6 acres ?

CHAPTER XX

RATIO AND PROPORTION

174. Methods of Comparing Magnitudes. When we wish to compare two magnitudes, there are two ways in which the comparison may be made.

(1) We may determine by how much the one exceeds the other. This result is found by **subtraction.**

(2) We may determine how many times the one contains the other. Here the result is found by **division.**

Thus, if one line is 6 inches in length and another is 18 inches, we may say that the second is 12 inches longer than the first, or that the second is three times as long as the first.

Neither method of comparison can be used, unless the magnitudes compared are of the same denomination, or can be changed into equivalent magnitudes of the same denomination.

Thus, we can compare 3 lb. and 10 lb.; 2 yd. 1 ft. and 2 ft. 9 in.; but we can not compare 5 lb. and 4 ft.

175. Ratio. When two magnitudes, of the same kind, are compared by division, the quotient is called the **ratio** of the magnitudes.

Thus, the ratio of 3 to 4 is the same as the quotient of $3 \div 4$, which is usually written $\frac{3}{4}$.

The ratio of 3 to 4 is written $3 : 4$,

$$\therefore \quad 3 : 4 = 3 \div 4 = \tfrac{3}{4}.$$

Similarly,

$$a : b = a \div b = \frac{a}{b}.$$

It will thus be seen that all problems in ratio may be considered as problems in fractions.

176. Comparison of Ratios. To compare two ratios we simply compare the fractions to which these ratios are equivalent.

Ex. 1.—Which is the greater ratio, 3 : 4 or 7 : 9 ?

The problem is at once changed into : " Which is the greater fraction $\frac{3}{4}$ or $\frac{7}{9}$? "

To compare the fractions we reduce them to the same denomination in the forms $\frac{27}{36}$ and $\frac{28}{36}$, and it is seen that the latter is the greater. We might also compare them by reducing the fractions to equivalent decimals.

Ex. 2.—Which is greater, $a : a+2$ or $a+1 : a+3$.

$$\frac{a}{a+2}=\frac{a(a+3)}{(a+2)(a+3)}=\frac{a^2+3a}{(a+2)(a+3)}.$$

$$\frac{a+1}{a+3}=\frac{(a+1)(a+2)}{(a+2)(a+3)}=\frac{a^2+3a+2}{(a+2)(a+3)}.$$

What is the conclusion ?

177. Terms of a Ratio. In the ratio $a : b$, a and b are called the **terms** of the ratio, a being called the **antecedent** and b the **consequent**. The antecedent corresponds to the numerator of the equivalent fraction, and the consequent to the denominator.

Thus, $\frac{a}{b}=\frac{\text{antecedent}}{\text{consequent}}=\frac{\text{numerator}}{\text{denominator}}=\frac{\text{dividend}}{\text{divisor}}.$

178. Equal Ratios. Since a ratio is a fraction, all the laws which we have used with fractions may also be used with ratios.

Thus, since $\frac{a}{b}=\frac{ma}{mb}$, it follows that $a : b=ma : mb$.

Hence *both terms of a ratio may be multiplied or divided by the same quantity (zero excepted) without changing the value of the ratio.*

Thus, $6 : 9=2 : 3$. $\frac{1}{2} : \frac{1}{3}=3 : 2$, $\frac{a}{b} : \frac{b}{a}=a^2 : b^2$

EXERCISE 123 (1-15, Oral)

Simplify the following ratios by expressing them as fractions in their lowest terms:

1. $10 : 15$.
2. $2\frac{1}{2} : 5$.
3. $45 : 63$.
4. $15 : 10$.
5. $\$2 : \$6$.
6. $\$2{\cdot}50 : \$10$.
7. 2 ft. : 3 yd.
8. 2 days : 12 hr.
9. 2 ft. 3 in. : 3 ft. 3 in.
10. $24a : 8a$.
11. $5ab : 10a^2$.
12. $a+b : a^2-b^2$.
13. $a-b : a^2-b^2$.
14. $x-y : x-y$.
15. $1-\frac{1}{x} : 1+\frac{1}{x}$.
16. $\frac{1}{x-2} : \frac{1}{x^2-5x+6}$.
17. $x^2+2xy+y^2 : x^2+y^2$.

18.* If 12 inches = 30·48 centimetres, find the ratio of an inch to a centimetre and of a metre to a yard.

19. The edges of two cubes are 2 inches and 3 inches. Find the ratios of their volumes.

20. If 25 francs = \$4·80, find the ratio of a franc to a dollar and of a quarter to a franc.

21. If a metre = 39·37 inches, find the ratio of a kilometre to a mile.

22. Which is the greater $2 : 3$ or $4 : 5$, $15 : 37$ or $11 : 27$, $a : a+2$ or $a+3 : a+5$?

23. Arrange in descending order of magnitude:

$$2 : 3,\ 3 : 5,\ 11 : 15,\ 13 : 18.$$

24. What is the effect of adding the same number 5 to both terms of the ratio $7 : 15$? What is the effect of subtracting 5 from each term?

25. What is the effect of adding 5 to each term of $15 : 7$? Of subtracting 5? Compare your results with the results of Ex. 24.

26. Separate 360 into three parts which are in the ratio $2 : 3 : 4$. (Let the parts be $2x$, $3x$, $4x$.)

27. Divide 165 into two parts in the ratio of $2 : 3$; 510 in the ratio $3 : 7$; 36 in the ratio $1\frac{1}{2} : 2\frac{1}{2}$.

28. When a sum of money is divided in the ratio 1 : 2, the smaller part is $20 more than when it is divided in the ratio 2 : 7. Find the sum.

29. What number must be added to both terms of [illegible] to make it equal to [illegible] ?

30. What number subtracted from each term of 7 : 10 will produce 13 : 19 ?

31. What must be added to each term of $a : b$ to produce $c : d$? What is the conclusion when $c = d$?

32. If a is a positive number which is the greater ratio,

$$\frac{1+2a}{1+3a} \text{ or } \frac{1+3a}{1+4a} ?$$

33. The rate of one train is 30 miles per hour and of another is 55 feet per second. What is the ratio of their rates ?

34. Divide a line a inches long into two parts whose lengths are in the ratio $b : c$.

35. A's income : B's income $= 3 : 4$, and A's expenditure : B's expenditure $= 5 : 6$. If A spends all his income, what per cent. of his income does B save ?

36. Divide $315 among A, B and C, so that A's share will be to B's as 3 : 4, and B's to C's as 5 : 7.

37. A line is divided into two parts in the ratio of 5 : 7 and into two parts in the ratio 3 : 5. If the distance between the points of division is 1 inch, find the length of the line.

38. Two numbers are in the ratio of 3 : 5, but if 10 be taken from the greater and added to the smaller, the ratio is reversed. Find the numbers.

39. Two bodies are moving at uniform rates. The first goes m feet in a seconds and the second n yards in b minutes. What is the ratio of their rates ?

17[illegible]. Proportion. A proportion is the statement of the equality of two ratios.

Thus, $3 : 4 = 15 : 20$, since $\frac{3}{4} = \frac{15}{20}$.

Therefore, $3 : 4 = 15 : 20$ is a **proportion**, or 3, 4, 15, 20 are said to be **in proportion**, or they are said to be **proportionals.**

If a, b, c, d are in proportion or $a:b=c:d$,

then
$$\frac{a}{b}=\frac{c}{d},$$
$$\therefore \quad ad=bc.$$

In the proportion $a:b=c:d$, a and d are called the **extremes** and b and c the **means.**

Since $ad=bc$, it is seen that *the product of the extremes is equal to the product of the means.*

180. Fourth Proportional. When $a:b=c:d$, d is called the **fourth proportional** to a, b, c.

Thus, if the fourth proportional to 10, 12, 15 is x,

then
$$10:12=15:x \text{ or } \frac{10}{12}=\frac{15}{x},$$
$$\therefore \quad 10x=12\times 15,$$
$$\therefore \quad x=18.$$

181. To find a Ratio, by Solving an Equation. From certain types of equations in x and y, the value of the ratio of $x:y$ may be found.

Ex. 1.—If $5x=6y$, find the ratio of $x:y$.

Since
$$5x=6y,$$
$$\therefore \quad x=\tfrac{6}{5}y,$$
$$\therefore \quad \frac{x}{y}=\frac{6}{5},$$
$\therefore$ the ratio of
$$x:y=\tfrac{6}{5} \text{ or } 6:5.$$

Ex. 2.—If $3x+4y=3y-7x$, find $\dfrac{x}{y}$.
$$3x+7x=3y-4y,$$
$$\therefore \quad \frac{x}{y}=-\frac{1}{10}.$$

If each term in the equation is of the first degree in x or y, the ratio of $x:y$ can be found, but it can not be found if there is a term not containing x or y.

Thus, from $2x-7y=10$, the value of $x:y$ can not be found.

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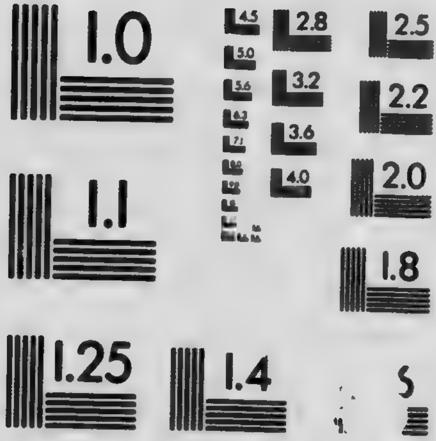
(ANSI and ISO TEST CHART No. 2)



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Ex. 3.—If $2x^2-7xy+6y^2=0$, find $x:y$.

Factoring, $(x-2y)(2x-3y)=0$,

$$\therefore \quad x-2y=0 \text{ or } 2x-3y=0,$$

$$\therefore \quad \frac{x}{y}=2 \text{ or } \frac{3}{2}.$$

Here there are two values of $\frac{x}{y}$. If we divide each term of the given equation by y^2, we get

$$2\left(\frac{x}{y}\right)^2-7\left(\frac{x}{y}\right)+6=0.$$

In this form we see that the equation is a quadratic in $\frac{x}{y}$, and we might naturally expect to find two values for the required ratio.

Ex. 4.—If

$$2x-5y+z=0,$$
$$3x+2y-2z=0,$$

find the ratios of x, y, z.

If we eliminate z in the usual way, we get

$$7x-8y=0,$$

$$\therefore \quad \frac{x}{y}=\frac{8}{7}, \qquad \therefore \quad \frac{x}{8}=\frac{y}{7}.$$

If we eliminate y we get

$$\frac{x}{z}=\frac{8}{19}, \qquad \therefore \quad \frac{x}{8}=\frac{z}{19}.$$

We can combine these results in the convenient form :

$$\frac{x}{8}=\frac{y}{7}=\frac{z}{19}.$$

EXERCISE 128 (1-21, Oral)

Find the value of x in the proportions :

1. $\frac{2}{3}=\frac{4}{x}$.

2. $\frac{3}{7}=\frac{9}{x}$.

3. $\frac{2}{5}=\frac{x}{3}$.

4. $\frac{x}{6}=\frac{5}{4}$.

5. $\frac{3}{x}=\frac{6}{-12}$

6. $\frac{x}{5}=-2$.

7. $\frac{a}{b}=\frac{c}{x}$.

8. $\frac{a}{b}=\frac{x}{c}$.

9. $\frac{4}{x}=\frac{x}{16}$.

10. $3:7=12:x$.

11. $2:3=x:9$.

12. $x:5=7:10$.

Find the value of $x : y$,

13. $2x=7y$. 14. $3y=12x$. 15. $2x-y=0$.

16. $\frac{1}{3}x=4y$. 17. $2x=-3y$. 18. $3y+11x=0$.

19. $x^2=4y^2$. 20. $4x^2=9y^2$. 21. $(x-3y)(x-5y)=0$.

22. If $\frac{a}{b}=\frac{c}{d}$, show that $\frac{a}{c}=\frac{b}{d}$, and $\frac{b}{a}=\frac{d}{c}$.

23. Find a fourth proportional to: 2, 3, 18; 5, -7, -10; $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$; a, b, c; a, $2b$, $3c$.

24. Find a fourth proportional to: $a-b$, $(a+b)^2$, a^2-b^2; and to a^2-3a+2, a^2-5a+6, a^2-5a+4.

25. What number must be added to each of the numbers 2, 4, 17, 25 so that the results will be proportionals? (Verify.)

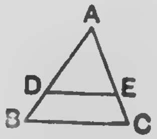
26. If $a+x$, $b+x$, $c+x$, $d+x$ are proportionals, find x. What does the result mean when $bc=ad$?

27. Find a in order that $a+3 : a+15=3 : 4$.

28. A's age is to B's as 4 : 5. Five years ago the ratio was 3 : 4. Find their ages.

29.* In an equilateral triangle the ratio of the altitude to either of the equal sides is $\sqrt{3} : 2$. If the altitude is 10 inches, find the side to two decimal places.

30. When a line is drawn parallel to the base of a triangle it divides the sides in the same ratio. In the figure, $AB=20$, $AD=14$ and $AC=15$. Find AE and EC.



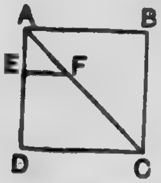
31. In the figure, the triangles ADE and ABC are similar. When triangles are similar their corresponding sides are in the same ratio, so that

$$AD : AB=DE : BC=AE : AC.$$

If $AB=8$, $BC=10$, $AC=9$, $AD=6$, find the lengths of all the other lines in the figure.

32. In the same figure, the areas of the similar triangles are in the same ratio as the areas of the squares on their corresponding sides. If $AD=20$ and $AB=35$, find the area of ADE if the area of ABC is 735.

33 The side of the square $ABCD$ is 10 inches and EF is parallel to DC. If the length of AE is 3 inches, find the length of FC to three decimals.



34. If the bases of two triangles are in the ratio 3 : 4 and their heights in ratio 8 : 9, find the ratio of their areas.

35. From these equations find $x : y$,

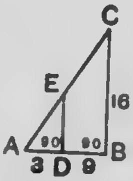
$$13x+5y=9x+13y;\quad ax+by=cx+dy;$$
$$mx-ny=nx+my;\quad px+qy=0.$$

36. Find two values of $x : y$ when

$$6x^2-13xy+6y^2=0;\quad x^2=4xy+5y^2.$$

37. If $5a-3b+2c=0$ and $a+b+c=0$, find the ratios of $a : b$, $a : c$, $a : b : c$.

38. Find the lengths of all the other lines in this figure.



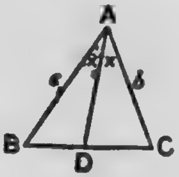
39. If a pole 10 feet high casts a shadow $17\frac{1}{2}$ feet long, what will be the length of the shadow cast, at the same time, by a monument 84 feet high?

40. Write the equation $3x^2-10xy+3y^2=0$ in a form showing $\frac{x}{y}$ as the unknown, and find $x : y$.

41. A number of two digits bears the ratio 7 : 4 to the number formed by reversing the digits. If the sum of the numbers is 66, find them.

42. The length of a room is to the width as 6 : 5, and the length is to the height as 3 : 2. If the area of the floor is $187\frac{1}{2}$ square feet, find the dimensions.

43. If 4 men and 3 women earn as much as 16 boys, and 6 men and 5 boys earn as much as 10 women, find the ratio of the earnings of a man, woman and boy.



44. If $3ab+2b^2 : 2a^2-ab=9 : 5$, find $a : b$.

45. When the angle A is bisected,

$$AB : AC=BD : DC.$$

(1) If $AB=10$, $AC=8$, $BC=12$, find BD and DC.

(2) If $AB=c$, $AC=b$, $BC=a$, find BD and DC.

46. The ratio of the area of a rectangle to the area of the square described on its diagonal is 6 : 13. Find the ratio of the sides.

47. The sides of a triangle are 7, 10 and 12. The perimeter of a similar triangle is 72½. What are its sides?

48. If $\frac{x+y}{5}=\frac{2x-3y}{7}=\frac{x+2y+5z}{9}$, find $x:y:z$.

182. Mean Proportional. When three numbers form a proportion, it is understood that the middle number is to be repeated. The three numbers are said to be in **continued proportion,** and the middle one is called the **mean proportional** between the other two.

Thus, 4, 6 and 9 are in continued proportion, since $4:6=6:9$. Here 6 is the mean proportional between 4 and 9.

If x is a mean proportional between 3 and 27,

$$\frac{3}{x}=\frac{x}{27}, \qquad \therefore \quad x^2=81, \qquad \therefore \quad x=\pm 9.$$

$\therefore$ the mean proportionals between 3 and 27 are ± 9.

Since $\frac{3}{9}=\frac{9}{27}$ and $\frac{3}{-9}=\frac{-9}{27}$, it is seen that these are the correct results.

Similarly, if x is the mean proportional between a and b, then

$$\frac{a}{x}=\frac{x}{b}, \qquad \therefore \quad x=\pm\sqrt{ab}.$$

Therefore, *the mean proportional between any two quantities is the square root of their product.*

183. Third Proportional. If a, b, c are in continued proportion, c is called the **third proportional** to a and b.

Thus, if x is the third proportional to 6 and 15,

$$\frac{6}{15}=\frac{15}{x}, \qquad \therefore \quad 6x=225, \qquad \therefore \quad x=37\tfrac{1}{2}.$$

EXERCISE 124

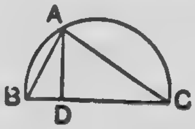
1. Find a mean proportional between 4 and 16; $2a$ and $3a$; $4ab^3$ and $9a^3b$; $(a-b)^2$ and $(a+b)^2$.

2. Find a third proportional to 2 and 4; 3 and 30; $5a$ and $10ab$; x^2-y^2 and $x-y$.

3. The mean proportional between two numbers whose sum is 34 is 15. Find the numbers.

4. Three numbers are in continued proportion. The middle one is 12 and the sum of the other two is 51. Find the numbers.

5. What number must be added to each of the numbers 3, 7, 12 so that the results will be in continued proportion?



6.* In the figure, the angle BAC being in a semicircle is a right angle. When AD is drawn perpendicular to the hypotenuse it is proven in geometry that

AD is a mean proportional between BD and DC,
AB between BD and BC, and AC between CD and BC

(1) If $BD=4$, $DC=9$, find AD.
(2) If $BD=5$, $AB=8$, find DC.
(3) If $BC=13$, $AC=12$, find DC, AB, AD.
(4) If $AB=3$, $AC=4$, find BC, AD, BD.

7. How would you use the preceding to find
(1) A line whose length is $\sqrt{6}$ inches?
(2) The side of a square whose area is 12 square inches?

8. Find two numbers such that the mean proportional between them is 4 and the third proportional to them is 32.

9. Divide a line 21 inches long into three parts such that the longest is four times the shortest and the middle one is a mean proportional between the other two.

184. The following examples will illustrate a method which has many appplications to problems with ratios or fractions.

Ex. 1.—If $\frac{a}{b}=\frac{c}{d}$, prove that $\frac{3a^2+2b^2}{a^2-5b^2}=\frac{3c^2+2d^2}{c^2-5d^2}$.

Since $\frac{a}{b}=\frac{c}{d}$, let each fraction $=k$.

Then $\frac{a}{b}=k$, $\therefore\ a=bk$, $\frac{c}{d}=k$, $\therefore\ c=dk$.

Substitute these values of a and c in each side of the identity to be proven.

$$\frac{3a^2+2b^2}{a^2-5b^2}=\frac{3b^2k^2+2b^2}{b^2k^2-5b^2}=\frac{b^2(3k^2+2)}{b^2(k^2-5)}=\frac{3k^2+2}{k^2-5}.$$

$$\frac{3c^2+2d^2}{c^2-5d^2}=\frac{3d^2k^2+2d^2}{d^2k^2-5d^2}=\frac{d^2(3k^2+2)}{d^2(k^2-5)}=\frac{3k^2+2}{k^2-5}.$$

$$\therefore\quad \frac{3a^2+2b^2}{a^2-5b^2}=\frac{3c^2+2d^2}{c^2-5d^2}.$$

Ex. 2.—If $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, prove $\frac{x^2y+y^2z+z^2x}{a^2b+b^2c+c^2a}=\frac{(x+y+z)^3}{(a+b+c)^3}$.

Let $x=ak$, $y=bk$, $z=ck$. Substitute as before and show that each of the fractions is equal to k^3.

185. **If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.**

Prove this by letting $a=bk$ and $c=dk$, as in the preceding examples.

Here the fraction $\frac{a+b}{a-b}$ was obtained by adding and subtracting the terms of the fraction $\frac{a}{b}$, and $\frac{c+d}{c-d}$ was obtained in a similar way from $\frac{c}{d}$.

This principle is sometimes useful in simplifying equations.

Ex. 1.—Solve $\frac{4x+3}{4x-3}=\frac{3a+4b}{3a-4b}$.

Adding and subtracting, $\frac{8x}{6}=\frac{6a}{8b}$,

$$\therefore\quad 64bx=36a,$$

$$\therefore\quad x=\frac{9a}{16b}.$$

Solve also, in the usual way, by cross multiplication.

Ex. 2.—If $\frac{a+b+c+d}{a-b-c+d}=\frac{a+b-c-d}{a-b+c-d}$, prove $\frac{a}{b}=\frac{d}{c}$.

Adding and subtracting, $\frac{2a+2d}{2b+2c}=\frac{2a-2d}{2b-2c}$,

$$\therefore\ \frac{a+d}{b+c}=\frac{a-d}{b-c},$$

$$\therefore\ \frac{a+d}{a-d}=\frac{b+c}{b-c}.$$

Adding and subtracting, $\frac{2a}{2d}=\frac{2b}{2c}$,

$$\therefore\ \frac{a}{d}=\frac{b}{c},$$

$$\therefore\ \frac{a}{b}=\frac{d}{c}.$$

EXERCISE 125

1. If $\frac{a}{b}=\frac{c}{d}$, prove that $\frac{a}{2a+3b}=\frac{c}{2c+3d}$.

2. If $a:b=c:d$, show that

$$ma+nb:ma-nb=mc+nd:mc-nd.$$

3. If $a:b=c:d$, prove $a^2bd+b^2c+bc=ab^2c+abd+ad$.

4.* If $\frac{x}{y}=\frac{3}{4}$, find the value of $\frac{2x^2+3xy}{6xy+y^2}$.

(Here $x=\frac{3}{4}y$, substitute for x and simplify.)

5. If $\frac{x}{2}=\frac{y}{3}$, find the value of $\frac{(2x+3y)(3x+2y)}{(5x-3y)(3x-5y)}$.

6. If $\frac{x}{y}=3$ and $\frac{a}{b}=\frac{2}{5}$, find the value of $\frac{12ax-by}{2ax+3by}$.

7. If $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, prove that each fraction is equal to $\frac{x+y+z}{a+b+c}$, that is, to $\frac{\text{sum of numerators}}{\text{sum of denominators}}$.

8. If $\frac{x}{b-c}=\frac{y}{c-a}=\frac{z}{a-b}$, prove $x(b+c)+y(c+a)+z(a+b)=0$

9. If a, b, c, d are proportionals, prove that a^2c+ac^2, b^2d+bd^2, a^2c^2 and b^2d^2 are proportionals.

10. If $\frac{a}{x+y}=\frac{b}{y-z}=\frac{c}{z+x}$, prove that $a=b+c$.

11. If $a:b=b:c$, show that $a:c=a^2:b^2$.

12. If $a-b:a+b=c-d:c+d$, prove $a:b=c:d$.

13. If $\frac{a+b+c+d}{a+b-c-d}=\frac{a-b+c-d}{a-b-c+d}$, show that $\frac{a}{b}=\frac{c}{d}$.

14. Solve $\frac{3x+4b}{3x-4b}=\frac{5a+3b}{5a-3b}$.

15. If the sum of two numbers is to their difference as 7 to 4, find the ratio of the numbers.

16. If $\frac{a-b+1}{b-c+2}=\frac{b-c+3}{c-a+4}=\frac{c-a+5}{a-b+6}$, show that each fraction equals $\frac{3}{4}$. (Use Ex. 7.)

17. Solve $\frac{ax+b+c}{ax-b+c}=\frac{bx+c+a}{bx+c-a}$.

18. If $a:b=3:5$, $b:c=7:9$, $c:d=15:16$, find the ratio of $a:d$.

19. If $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, show that each fraction equals $\frac{5x-3y+2z}{5a-3b+2c}$, and also equals $\frac{mx+ny-pz}{ma+nb-pc}$.

20. Find two numbers such that their sum, difference and product are proportional to 4, 2, 9.

21. If a, b, c are consecutive numbers and if $c^2-b^2:b^2-a^2=41:39$, find the numbers.

22. The length and breadth of a room are as $3:2$, and if 2 feet be added to each, the new area of the floor is to the old as $35:27$. Find the dimensions.

23. If $a:b=c:d$, prove $a:a+b=a+c:a+b+c+d$.

24. If $\frac{10a+b}{10c+d}=\frac{12a+b}{12c+d}$, show that $\frac{a}{b}=\frac{c}{d}$.

25. If $a:b=b:c$, then $a^2+ab:b^2=b^2+bc:c^2$.

EXERCISE 103 (Review of Chapter XX)

Write as fractions in their simplest forms :

1. $7\frac{1}{2} : 8\frac{1}{3}$.
2. $x^2-y^2 : (x-y)^2$.
3. $a^2+b^2 : (a+b)^2$.
4. $1-\frac{1}{x^2} : 1+\frac{1}{x}$.
5. $a^2-\frac{b^2}{a} : 1-\frac{b}{a}$.
6. $a-\frac{b}{c} : c-\frac{b}{a}$.
7. $\frac{1}{x^2-5x+6} : \frac{1}{x^2+x-12}$.
8. $a^2+1+\frac{1}{a^2} : a-1+\frac{1}{a}$.

9. Divide 144 into three parts proportional to 3, 4, 11.

10. What must be added to each term of 4 : 7 to make it equal to 6 : 7 ?

11. Write as a proportion in two ways :

$$3 \cdot 6=2 \cdot 9\ ;\ 2 \cdot 5=3x\ ;\ ab=cd\ ;$$
$$(a+b)(a-b)=3 \cdot 4\ ;\ a^2-5a+6=a^2+5a+4.$$

12. If the means are 7 and 12 and one extreme is 3, what is the other extreme ?

13. Find a fourth proportional to : 7, 15, 35 ;

$$a,\ a^2,\ a^3\ ;\ x+y,\ x-y,\ x^2-y^2\ ;\ \frac{1}{a-b},\ \frac{1}{a+b},\ a^2-b^2.$$

14. Find two numbers in the ratio 9 : 5, the difference of whose squares is 504.

15. Two numbers are in the ratio of 5 : 8, and if 8 be added to the less and 2 be taken from the greater, the ratio is 14 : 15. Find the numbers.

16. Find two numbers in the ratio 6 : 5 so that their sum is to the difference of their squares as 1 : 3.

17.* If the ratio $a-x : b-x$ is equal to the square of the ratio $a : b$, find x.

18. If $2x+3y : 3x-5y=9 : 11$, find $x : y$.

19. If $(5x-7y)(2x-3y)=(4x-5y)(x-y)$, find $x : y$.

20. If $4x-5y=2x+2y$, find $3x+2y : 2x+3y$.

21. If $6x^2+15y^2=19xy$, find $x : y$.

22. If $x^2+x+1 : 62(x+1)=x^2-x+1 : 63(x-1)$, find x.

23. If $2x+y-2z=0$ and $7x+6y-9z=0$, find $x : y$, $x : z$ and $x : y : z$

24. If $\frac{x}{y}=\frac{2}{3}$, find the value of $\frac{6x-2y}{3x+11y}$.

25. Find a mean proportional to $x^2 - \frac{1}{y^2}$ and $y^2 - \frac{1}{x^2}$.

26. If $ax+by : bx+ay = 9 : 11$ and $a : b = 3 : 2$, find the ratio of x to y.

27. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, show that each of these fractions is equal to $\frac{ma-nc-pe}{mb-nd-pf}$.

28. Find two numbers whose sum, difference and product are proportional to 5, 3, 16.

29. If $a : b = c : d$, show that $\frac{a}{c} = \frac{a+b}{c+d}$; $\frac{2a^2-3b^2}{2a^2+3b^2} = \frac{2c^2-3d^2}{2c^2+3d^2}$;

$$(ab+cd)^2 = (a^2+c^2)(b^2+d^2).$$

30. If $\frac{x}{b+c-a} = \frac{y}{c+a-b} = \frac{z}{a+b-c}$, prove that

$$x(b-c)+y(c-a)+z(a-b)=0.$$

31. If any number of ratios are equal, show that each ratio is equal to the ratio of the sum of all the antecedents to the sum of all the consequents.

32. If $3x-2y+4z = 2x-3y+z = 0$, find the ratios of x, y, z. If also, $x^2+y^2+z^2=150$, find the values of x, y and z.

33. The hypotenuse of a right-angled triangle is to the shortest side as 13 : 5. If the perimeter is 120, find the sides.

34. The length, width and height of a room are proportional to 4, 3, 2. If each dimension be increased 2 feet, the area of the four walls will be increased in the ratio of 10 to 7. Find the dimensions of the room.

35. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, show that $\frac{a^3+c^3+e^3}{b^3+d^3+f^3} = \frac{ace}{bdf}$.

36. If the sides of a triangle are 6 and 8 and the base is $4\frac{2}{3}$, find the segments of the base when the bisector of the vertical angle is drawn.

37. If $\frac{y-z+x}{z-x+y} = \frac{x-y+z}{x+y+z}$, show that $z^2=x^2+y^2$.

38. The incomes of A and B are as 2 : 3 and their expenses are as 6 : 7. If A saves 25% of his income, what % does B save?

39. Find three values of the ratio $x : y$ if

$$3(x^3-4x^2y+5xy^2-2y^3) = 2(x^3-2x^2y-2xy^2+3y^3).$$

CHAPTER XXI

THE GENERAL QUADRATIC EQUATION

186. Type of the General Quadratic. The equation

$$ax^2+bx+c=0,$$

is called the general quadratic equation, because every quadratic equation may be reduced to this form.

If the factors of ax^2+bx+c can be obtained, the roots of the equation can be found by solving the two equivalent equations.

187. Solution of Literal Quadratics. The method of completing the square may be applied to the solution of quadratic equations with literal coefficients.

Ex. 1.—Solve $\qquad x^2+2mx=n.$

Complete the square by adding m^2 to each side,

$$\therefore \quad x^2+2mx+m^2=n+m^2.$$

Take the square root, $\qquad x+m=\pm\sqrt{n+m^2},$

$$\therefore \quad x=-m\pm\sqrt{n+m^2}.$$

The two roots are $-m+\sqrt{n+m^2}$, $-m-\sqrt{n+m^2}$.

Ex. 2.—Solve $\qquad x^2+px+q=0.$

Transpose the absolute term, $x^2+px=-q$.

Add $\frac{p^2}{4}$ to each side, $x^2+px+\frac{p^2}{4}=-q+\frac{p^2}{4}=\frac{p^2-4q}{4}.$

Take the square root, $\qquad x+\frac{p}{2}=\pm\frac{\sqrt{p^2-4q}}{2},$

$$\therefore \quad x=\frac{-p}{2}\pm\frac{\sqrt{p^2-4q}}{2}.$$

Ex. 3.—Solve $ax^2+bx+c=0.$

Divide by a to make the first term a square,

$$\therefore\quad x^2+\frac{b}{a}x+\frac{c}{a}=0.$$

Transpose,
$$x^2+\frac{b}{a}x=-\frac{c}{a}.$$

Add $\frac{b^2}{4a^2}$ to each,
$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a}=\frac{b^2-4ac}{4a^2}.$$

Take the square root,
$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a},$$

$$\therefore\quad x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a},$$

$$=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

The roots of the general quadratic equation are

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

188. The roots of the general quadratic might also be found by factoring as in art. 171.

$$ax^2+bx+c=0,$$

$$\therefore\quad a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)=0,$$

$$\therefore\quad a\left\{\left(x+\frac{b}{2a}\right)^2+\frac{c}{a}-\frac{b^2}{4a^2}\right\}=0,$$

$$\therefore\quad a\left\{\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right\}=0$$

$$\therefore\quad a\left(x+\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\right)\left(x+\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\right\}=0.$$

Since the product is zero, one of the factors must be zero. But a is not zero, as the equation would not then be a quadratic.

$$\therefore\quad x+\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}=0\text{ or }x+\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}=0,$$

$$\therefore\quad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

EXERCISE 127

Solve by either of the preceding methods:

1.* $x^2-2ax=3a^2$.
2. $x^2+4bx-5b^2=0$.
3. $x^2-6mx+3m^2=0$.
4. $x^2+4px-p^2=0$.
5. $x^2-2ax+b=0$.
6. $x^2+2bx-c=0$.
7. $ax^2+2ax=b$.
8. $ax^2+2bx+c=0$.
9. $ax^2-bx-c=0$.
10. $px^2-qx+r=0$.

189. Solving by Formula. The roots of any particular quadratic equation may be found by substituting the values of a, b and c in the roots of the general quadratic.

Ex. 1.—Solve $6x^2-7x+2=0$.

Here $a=6$, $b=-7$, $c=2$.

Substitute these values in $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$,

$$\therefore\quad x=\frac{+7\pm\sqrt{49-48}}{12},$$

$$=\frac{7\pm1}{12}=\frac{8}{12}\text{ or }\frac{6}{12}=\frac{2}{3}\text{ or }\frac{1}{2}$$

Verify by substitution.

Ex. 2.—Solve $5x^2+6x-1=0$.

Here $a=5$, $b=6$, $c=-1$.

$$\therefore\quad x=\frac{-6\pm\sqrt{36-(-20)}}{10}=\frac{-6\pm\sqrt{56}}{10},$$

$$=\frac{-6\pm2\sqrt{14}}{10}=\frac{-3\pm\sqrt{14}}{5}.$$

In this case the roots are irrational, but, if necessary, we may substitute for $\sqrt{14}$ its approximate value 3·742, when the roots become

$$\frac{-3\pm3{\cdot}742}{5}=\frac{{\cdot}742}{5}\text{ or }\frac{-6{\cdot}742}{5}={\cdot}148\text{ or }-1{\cdot}348.$$

NOTE.—The pupil is warned to be careful of the signs when substituting, particularly when c is negative.

Ex. 3.—Solve $2x^2-5x+6=0.$

$$a=2,\ b=-5,\ c=6.$$

$$\therefore\quad x=\frac{+5\pm\sqrt{25-48}}{4}=\frac{5\pm\sqrt{-23}}{4}.$$

190. Imaginary Roots. In the preceding result the numerical value of the roots cannot be found even approximately, for there is no number whose square is negative.

Such a quantity as $\sqrt{-23}$ is called an **imaginary quantity,** and the roots in this case are said to be **imaginary.** This is merely another way of saying that there is no real number which will satisfy the equation $2x^2-5x+6=0$.

191. Methods of Solving Quadratic Equations. When a quadratic equation has been reduced to the standard form, it may be solved:

(1) *By factoring, by inspection or by completing the square.*

(2) *By substitution in the general formula.*

The pupil is advised to try to factor by inspection, and if this method is unsuccessful, then substitute in the general formula.

As the general formula will be used very frequently, it is absolutely essential that it be committed to memory.

The roots of $ax^2+bx+c=0$ **are** $\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$.

EXERCISE 128

Solve, using the formula:

1.* $3x^2-5x+2=0.$

2. $24x^2-46x+21=0.$

3. $575x^2-2x=1.$

4. $2x^2-6x-1=0.$

5. $247x^2+5x=12.$

6. $2x^2-13x+10=0.$

7. $391x^2+4x=35.$

8. $1200x^2-10x=1.$

9. $x^2+x(3b-2a)=6ab.$

10. $2x^2-25x+77=0.$

11. $6x^2-x-1=0.$

12. $1800x^2-5x-1=0.$

Solve by any method. Verify 13–18:

13. $27x^2-24x=16.$ **14.** $15x^2+7x-2=0.$ **15.** $12x^2-x-6=0$

16. $4x^2-17x+4=0.$ **17.** $460x^2-3x=1.$ **18.** $5-26x+5x^2=0$

19. $9x+4=5x^2.$ **20.** $3x^2+2=9x.$ **21.** $2x^2-2x=\frac{3}{2}.$

22. $4x^2-4x=79.$ **23.** $\frac{3}{4}-y=y^2.$ **24.** $\frac{1}{3}+\frac{x}{9}=\frac{2}{x}.$

25. $x-\frac{3}{x+2}=0.$ **26.** $\frac{3y}{2}+\frac{1}{2}=\frac{1}{3y}.$ **27.** $x^2-\frac{7}{12}x=1.$

28. $(x-4)^2-3(x-9)=15.$ **29.** $(x-2)(x+3)=x(5x-9)-2.$

30. $2ax^2+x(a-2)=1.$ **31.** $ac+\frac{b}{a}=\frac{b}{x}+cx.$

32. $2x(x-2)=a^2-2.$ **33.** $\frac{x}{2}+\frac{2}{x}=\frac{x}{3}+\frac{3}{x}.$

34. $\frac{4}{x+1}+\frac{5}{x+2}=\frac{12}{x+3}.$ **35.** $(x+2)^2+(x+3)^2=(x+6)^2.$

36. $x^2-xy-3y^2=-12.$ If $y=2$, find x.

37. $x^2-4xy+x^2+y^2+5=0.$ If $x=-3$, find y.

38. If $\frac{x}{x+1}=\frac{x+1}{2x}$, find x to three decimal places.

39. Find the sum of the roots of $x^2-3x=20.$

40. The area of a square in square feet and its perimeter in inches are expressed by the same number. Find the side of the square.

41. The length of a rectangular field exceeds the width by 16 rods and the area is 32 acres. Find the length.

42. Find three consecutive even numbers whose sum is $\frac{1}{8}$ of the product of the first two.

43. A line 10 inches long is divided into two segments so that the square on the longer segment is equal to the rectangle contained by the whole line and the shorter segment. Find the segments to two decimal places.

44. Find two numbers whose difference is 3 and the sum of whose squares is 317.

45. The area of a square is doubled by adding 5 inches to one side and 12 inches to the other. Find the side of the square.

46. Three times the square of a number exceeds eight times the number by unity. Find the number to three decimals.

47. Mr. Gladstone was born in the year A.D. 1809. In the year A.D. x^2 he was $x-3$ years old. Find x.

48. The area of a rectangular field is half an acre. The perimeter is 201 yards. Find the sides.

49. One root of $x^2-5x+d=0$ is 8. Find the value of d and the other root.

50. If a train travels 10 miles per hour faster than its usual rate, it will cover 480 miles in 4 hours less time. Find its usual rate.

51. Divide 3 into two parts so that the sum of their squares may be $4\frac{5}{9}$.

52. I buy a number of articles for \$4·80 and sell for \$5·95 all of them but 2 at 6 cents a dozen more than they cost. How many did I buy?

53. A straight line AB, 12 inches in length, is divided at C so as to satisfy one of the following conditions. Find, in each case, the length of AC to two decimals:

(1) $AC^2=2BC^2$.
(2) $AC^2=2AB\,.\,BC$.
(3) $3AC^2=4AB\,.\,BC$.
(4) $AC^2+3BC^2=2AB^2$.
(5) $AC^2-BC^2=10$ sq. in.
(6) $AC(AB+BC)=2$ sq. ft.

54. I buy a number of books for \$6, the price being uniform. If they had been subject to a discount of 5 cents each, I could have bought 6 more for the same money. What did each cost?

55. Solve the equation $ax^2+bx+c=0$ by multiplying by $4a$ and completing the square of $2ax+b$.

56. Solve $\dfrac{3+2x}{2-x}-\dfrac{2-3x}{2+x}+\dfrac{16x-x^2}{x^2-4}=\dfrac{1}{3}$.

Verify the roots obtained.

192. Equations Solved like Quadratics. There are certain types of equations of a higher degree than the second, which may be solved by reducing them to the form of quadratics.

Ex. 1.—Solve $x^4-10x^2+9=0$.

This is an equation of the fourth degree, but we might write it i the form of a quadratic, thus

$$(x^2)^2-10(x^2)+9=0,$$

or if we write y for x^2 it takes the form

$$y^2-10y+9=0,$$

$$\therefore\ (y-9)(y-1)=0,$$

$$\therefore\ y=9 \text{ or } 1.$$

But $y=x^2$, $\quad\therefore\ x^2=9 \text{ or } 1,$

$$\therefore\ x=\pm 3 \text{ or } \pm 1.$$

We see that this equation has four roots. This is what we might expect, as it is an equation of the fourth degree in x.

Verify each of the four roots.

Ex. 2.—Solve $(x^2-5x)^2+4(x^2-5x)-12=0$.

Here we consider x^2-5x as the unknown, whose value should first be found.

Let $\quad x^2-5x=y,$

$$\therefore\ y^2+4y-12=0,$$

$$\therefore\ (y+6)(y-2)=0,$$

$$\therefore\ y=-6 \text{ or } 2.$$

$$\therefore\ x^2-5x=-6,$$

$$\therefore\ x^2-5x+6=0,$$

$$\therefore\ (x-3)(x-2)=0,$$

$$\therefore\ x=3 \text{ or } 2.$$

or $\quad x^2-5x=2,$

$$\therefore\ x^2-5x-2=0,$$

$$\therefore\ x=\frac{5\pm\sqrt{25+8}}{2},$$

$$\therefore\ x=\frac{5\pm\sqrt{33}}{2}.$$

This equation has four roots, two of which are rational and the other two irrational.

Verify the rational roots.

Ex. 3.—Solve $(2x^2+3x-1)(2x^2+3x-2)=56$.

Let $\quad 2x^2+3x=y.$

The result is $x=\frac{3}{2},\ -3,\ \frac{-3\pm\sqrt{-39}}{4}$.

Ex. 4.—Solve $\dfrac{x^2+2x}{3}+\dfrac{3}{x^2+2x}=\dfrac{26}{5}$.

Let $\dfrac{x^2+2x}{3}=y, \quad \therefore \quad \dfrac{3}{x^2+2x}=\dfrac{1}{y},$

$$\therefore \quad y+\frac{1}{y}=\frac{26}{5}.$$

Complete Ex.'s 3 and 4 and verify the rational roots.

Ex. 5.—Solve $x^3-1=0.$

Factoring,

$$(x-1)(x^2+x+1)=0,$$

$$\therefore \quad x-1=0 \text{ or } x^2+x+1=0,$$

$$\therefore \quad x=1 \text{ or } x=\frac{-1\pm\sqrt{-3}}{2}.$$

We thus see that *if one root of an equation of the third degree, or a cubic equation, can be found by factoring, the equation can be completely solved.*

This equation might be written $x^3=1$, and each of the three roots when cubed must give unity, which shows that unity has three cube roots. This is what we might have expected, as we have already seen that unity has two square roots, $+1$ and -1.

EXERCISE 129

Solve and verify the rational roots:

1. $x^4-5x^2+4=0.$
2. $x^4-13x^2+36=0.$
3. $9y^4+12=31y^2.$
4. $8x^6-65x^3+8=0.$
5. $(x^2+5x+6)(x^2-9x+14)=0.$
6. $\dfrac{x^2+16}{25}+\dfrac{25}{x^2+16}=2.$
7. $(x^2-4x+5)(x^2-4x+2)=-2.$
8. $x^2+x+1=\dfrac{42}{x^2+x}.$
9. $(x^2+x+1)^2-4(x^2+x+1)+3=0.$
10. $x^3-4x^2-4x+16=0.$
11. $6\left(x+\dfrac{1}{x}\right)^2-35\left(x+\dfrac{1}{x}\right)+50=0.$
12. $(1+x+x^2)(x+x^2)=156.$

13.* $(x+1)(x+2)(x+3)(x+4)=120$. (Multiply the first and last factors and the second and third.)

14. $x(x-1)(x-2)(x-3)=360$.

15. Find the three cube roots of 8 by solving the equation $x^3-8=0$.

16. Find the four fourth roots of 16 by solving the equation $x^4-16=0$.

17. Solve $x^3-19x+30=0$ being given that 3 is one of the roots.

18. Solve $12x^3-29x^2+23x-6=0$ (use the factor theorem).

19. It is evident that 4 is a root of the equation

$$x(x-1)(x-2)=4\,.\,3\,.\,2.$$

Find the other two roots.

20. Find the six roots of $8x^6-217x^3+27=0$.

21. Solve $(x^2-x)^2-8(x^2-x)+12=0$.

22. Solve $x^2+\dfrac{1}{x^2}+x+\dfrac{1}{x}=4$ $\left(\text{Add to } x^2+\dfrac{1}{x^2}\right.$ the quantity required to make it the square of $\left.x+\dfrac{1}{x}.\right)$

EXERCISE 180 (Review of Chapter XXI)

1. Explain the different methods of solving quadratic equations. Illustrate them, by solving in full the equation $3x^2-4x-15=0$, by each method.

2. Solve $323x^2+2x=1$.

3. The difference of two numbers is 8 and the sum of their squares is 104. Find the numbers.

4. If $x=2\left(1+\dfrac{1}{x}\right)$, find x to two decimal places.

5. What is the price of meat per lb. if a reduction of 20% in the price would mean that 5 lb. more than before can be bought for \$3 ?

6. Solve $10x^2-19x-9=0$.

7. The sides of a right-angled triangle are a, $a-10$ and $a+10$ What are the sides ?

8. Solve $\frac{x-1}{x-4}+\frac{x-5}{x-6}=4$.

9. The sum of two numbers is 45 and the sum of their reciprocals is ·09. Find the numbers.

10. Solve $6375x^2-10x=1$.

11. The length of a rectangular field is 5 rods more than the width. The area is $3\frac{1}{8}$ acres. Find the sides.

12. What must be the values of n in order that $\frac{16a+2n^2}{10n+21a}$ may equal $\frac{3}{8}$ when $a=\frac{1}{25}$?

13. The perimeter of a rectangle is 56 and the area is 192. Find the diameter of the circle which passes through its angular points.

14. Solve $\cdot0075x^2+\cdot75x=150$.

15. By solving $(x-2)(x-3)=(a-2)(a-3)$, find a quantity which can be substituted for a in $(a-2)(a-3)$ without changing its value.

16. Solve $x^3-2x^2-89x+90=0$.

17. Two trains each run 330 miles. One of them, whose average speed is 5 miles per hour greater than the other, takes ½ an hour less to travel the distance. Find their average speeds.

18. Solve $\frac{x^2+2}{x+1}+\frac{x+1}{x^2+2}=2\frac{1}{2}$.

19. Solve $\frac{x}{x^2+1}+\frac{x^2+1}{x}=\frac{109}{30}$.

20. I sell a horse for \$96 and gain as much % as the horse cost in dollars. What was the cost?

21. Solve $(x^2-3x-5)^2+8(x^2-3x-5)+7=0$.

22. Divide 25 into two parts so that the sum of the fractions formed by dividing each part by the other may be 4 25.

23. The sides of a rectangular field are $x+17$ and $x-17$. The diagonal is 50. Find the area.

24. Solve $x^2+x+\frac{72}{x^2+x}=18$.

25.* Solve $(m^2-n^2)x^2+2x(m^2+n^2)+m^2-n^2=0$.

26. Solve $(x-2)(x-1)(x+2)(x+3)=60$.

27. Find all the roots of the equation $x^3=125$.

28. Since $x^2-8x+12=(x-2)(x-6)$, for what values of x will the expression $x^2-8x+12$ be equal to zero, and for what values will it be negative?

29. Solve $\frac{a}{x-a}-\frac{b}{x-b}=\frac{a-b}{a+b}$.

30. Solve $adx-acx^2=bcx-bd$.

31. The area of a square is trebled by adding 10 inches to one side and 12 inches to the other. Find the side of the square.

32. Solve $x^2(a^2-c^2)-x(ab+3bc)-2b^2=0$.

33. Solve $(x^2+6x+8)^2+3x(x^2+6x+8)=0$.

34. A man bought a number of acres for $300. If he had paid $5 more per acre, the number of acres would have been 2 less. Find the number bought.

35. Solve $\frac{1}{x+a+b}=\frac{1}{x}+\frac{1}{a}+\frac{1}{b}$.

36. Solve $\frac{x-a}{b}+\frac{x-b}{a}=\frac{b}{x-a}+\frac{a}{x-b}$.

37. *OX* and *OY* are two roads at right angles. *A* starts at noon along *OX* at 3 miles per hour. *B* starts at 2 o'clock along *OY* at 4 miles per hour. Find to the nearest minute when they will be 20 miles apart.

38. Solve $a^2x^2-2a^3x+a^4-1=0$.

39. Solve $ax^2-\frac{6c^2}{a+b}=cx-bx^2$.

40. A gravel path 2 yards wide is made round a square field and it is found that it takes up $\frac{1}{16}$ of the area of the field. Find the area of the field in square yards.

41. Solve $s=vt+16t^2$ for t.

42. What positive integer is that, the sum of whose square and cube is nine times the next higher integer?

43. Solve $(x^2+x-2)^2-4(x^2+x-2)+3=0$.

44. The side of a square is 34 inches. Find at what points in the sides the vertices of an inscribed square must be placed so that it may have an area of 676 square inches.

45. Write the equation $ax^2+bxy+cy^2=0$ as a quadratic in $\frac{x}{y}$. What are the values of $\frac{x}{y}$ and of $\frac{y}{x}$?

46. What positive integral value of x will make x^2+10x most nearly equal to 1000?

CHAPTER XXII

SIMULTANEOUS QUADRATICS

193. Consider the problem: The sum of two numbers is 12 and the sum of their squares is 74. Find the numbers.

Let $x =$ one of the numbers,

$\therefore\quad 12 - x =$ the other,

$$\therefore\quad x^2 + (12 - x)^2 = 74.$$

Solve this equation and find $x = 7$ or 5.

If $x = 7$ or 5, then $12 - x = 5$ or 7.

$\therefore$ the numbers are 5 and 7.

Here we have used only one unknown. We might have solved by using two unknowns.

Let x and y be the numbers,

$$\therefore\quad x + y = 12,$$

and

$$x^2 + y^2 = 74.$$

How can we obtain from these two equations the original equation in the preceding solution?

194. Type I.

Ex. 1.—Solve

$$x + 3y = 10, \qquad (1)$$

$$x^2 + xy = 4. \qquad (2)$$

From (1),

$$x = 10 - 3y. \qquad (3)$$

Substitute in (2),

$$(10 - 3y)^2 + y(10 - 3y) = 4,$$

$$\therefore\quad 100 - 60y + 9y^2 + 10y - 3y^2 = 4,$$

$$\therefore\quad 6y^2 - 50y + 96 = 0,$$

$$\therefore\quad 3y^2 - 25y + 48 = 0,$$

$$\therefore\quad (y - 3)(3y - 16) = 0,$$

$$\therefore\quad y = 3 \text{ or } \tfrac{16}{3}.$$

Substitute $y=3$ in (3) and $x=1$

" $y=\frac{16}{3}$ " " " $x=-6$

There are therefore two solutions,

$$x=1,\ y=3 \text{ or } x=-6,\ y=5\tfrac{1}{3}.$$

$$x=1 \text{ or } -6.$$

$$y=3 \text{ or } 5\tfrac{1}{3}.$$

Verify by showing that $x=1$, $y=3$ satisfies *both* equations and also $x=-6$, $y=5\frac{1}{3}$.

The pupil must note that $x=1$ was obtained from $y=3$, not from $y=5\frac{1}{3}$.

Therefore, $x=1$, $y=5\frac{1}{3}$ is not a solution, nor is $x=-6$, $y=3$. Verify this by substitution.

Equation (1) is a linear equation, or an equation of the first degree in x and y. Equation (2) is a quadratic equation, or an equation of the second degree in x and y.

A system of equations of this type, that is, where one is of the first degree and the other of the second degree, may always be solved by the method of substitution, which does not differ from the similar method employed in art. 107, when both equations were of the first degree.

Ex. 2.—Solve
$$3x-y=5, \qquad (1)$$
$$x^2+3xy=15. \qquad (2)$$

From (1), $y=3x-5$, $\therefore\ x^2+3x(3x-5)=15$,

$$\therefore\ 10x^2-15x-15=0,$$

$$\therefore\ 2x^2-3x-3=0,$$

$$\therefore\ x=\frac{3\pm\sqrt{33}}{4}=\frac{3\pm5{\cdot}745}{4}=2{\cdot}186 \text{ or } -{\cdot}686.$$

$$\therefore\ y=3x-5=\frac{-11\pm3\sqrt{33}}{4}=1{\cdot}558 \text{ or } -7{\cdot}058.$$

Here the roots are irrational and it is customary to leave them in that form, unless the decimal form is asked for.

EXERCISE 181

Solve and verify 1-6:

1. $x+y=7,$ $xy=12.$

2. $x-y=4,$ $xy=60.$

3. $x-2y=0,$ $x^2-y^2=27.$

4. $x-y=3,$ $x^2+y^2=65.$

5. $x-y=6,$ $x^2-y^2=60.$

6. $2x+y=9,$ $x^2-y^2=15.$

7.* $x+3y=11,$ $x^2+y=27.$

8. $2x+3y=12,$ $x^2+y^2=13.$

9. $3x-4y=2,$ $3x^2+2y^2=140.$

10. $x^2+3xy+y^2+2x=37$ $x-y=3.$

11. $3x^2-2xy+5x-y=17$ $2x-3y=1$

12. If $x-3y=2$ and $x^2-xy+2y^2=6$, find the values of x and y to three decimal places.

13. The hypotenuse of a right-angled triangle is 25 and the perimeter is 56. Find the sides.

14. *A* is 10 years older than *B*. Eight years ago the sum of the squares of the numbers representing their ages was 148. Find their ages.

15. The diagonal of a rectangle is 50. The difference of the sides is 10. Find the area.

16. The area of a right-angled triangle is 96 and the difference of the two sides about the right angle is 4. Find the hypotenuse.

17. Solve $3x+5y=2$, $3x^2-10y^2-xy+28=0$.

18. If each digit of a number be increased by 2, the product of these increased digits will be the original number. When the digits are interchanged the resulting number is thirteen times the tens digit of the original number. Find the number.

19. The sum of the areas of two squares is 40 square inches. The side of the smaller is 10 inches less than three times the side of the larger. Find their sides to three decimals.

20. Solve $\frac{x^2}{y^2}+5\frac{x}{y}=14$, $y-1=x$.

195. When both equations are of the second degree in x and y, they can not always be solved by elementary methods.

There are special cases in which they can be solved without difficulty.

196. Type II.

Solve $$x^2-5xy+4y^2=0, \qquad (1)$$
$$x^2+y^2+3x=29. \qquad (2)$$

Factoring (1), $$(x-4y)(x-y)=0,$$
$$\therefore \quad x=4y \text{ or } x=y.$$

We are now required to solve:

$$\left.\begin{matrix} x^2+y^2+3x=29 \\ x=4y \end{matrix}\right\} \text{ and } \left.\begin{matrix} x^2+y^2+3x=29 \\ x=y \end{matrix}\right\}.$$

Substituting the value of x,

$$\begin{array}{l|l} \therefore \quad 16y^2+y^2+12y=29, & \therefore \quad y^2+y^2+3y=29, \\ \therefore \quad 17y^2+12y-29=0, & \therefore \quad 2y^2+3y-29=0, \\ \therefore \quad (y-1)(17y+29)=0, & \therefore \quad y=\dfrac{-3\pm\sqrt{241}}{4}, \\ \therefore \quad y=1 \text{ or } -\frac{29}{17}, & \\ \therefore \quad x=4 \text{ or } -\frac{116}{17}. & \therefore \quad x=\dfrac{-3\pm\sqrt{241}}{4}. \end{array}$$

Here there are four solutions:

$$x=4 \text{ or } -\tfrac{116}{17} \text{ or } \frac{-3\pm\sqrt{241}}{4},$$

$$y=1 \text{ or } -\tfrac{29}{17} \text{ or } \frac{-3\pm\sqrt{241}}{4}.$$

In this type the first equation contains only terms of the second degree. When that is the case the left-hand member may be factored and each of the resulting linear equations may be combined with the second equation, thus giving two cases of Type I.

EXERCISE 182

Solve and verify 1–5:

1. $x^2-y^2=0,$
   $x^2+xy+y^2=36.$

2. $x^2-4xy+3y^2=0,$
   $x^2+y^2=10.$

3. $3x^2-2xy-y^2=0,$
   $x+y+y^2=32.$

4. $x^2+y^2+2x=12,$
   $3x^2+2xy=y^2.$

5. $4x^2+20xy+9y^2=0,$
$$2xy+1=0.$$

6.* $\dfrac{x^2}{y^2}+\dfrac{5x}{y}=14,$
$$x-1=y^2.$$

7. $6x^2-17xy+12y^2=0,$
$$x^2-xy-y=1.$$

8. $x^2+2y=5,$
$$6x^2+4y^2=11xy.$$

9. Find four solutions of the equations
$$(x-y)(x-2)=0,\ (x+y-6)(y+3)=0.$$

197. Type III. Homogeneous Equations.

Solve
$$x^2-xy=6, \qquad (1)$$
$$y^2+3xy=10. \qquad (2)$$

Multiply (1) by 5 and (2) by 3 and subtract, to eliminate the absolute terms, and we get
$$5x^2-14xy-3y^2=0. \qquad)$$

This equation (3) is of the same form as the first equation in Type II. Grouping (3) with (1) we proceed as before.

Factoring (3), $$(x-3y)(5x+y)=0,$$
$$\therefore \quad x=3y \text{ or } -\tfrac{1}{5}y.$$

Substitute $x=3y$ in (1),
$$\therefore \quad 9y^2-3y^2=6,$$
$$\therefore \quad y^2=1,$$
$$\therefore \quad y=\pm 1,$$
$$\therefore \quad x=\pm 3.$$

Substitute $x=-\frac{1}{5}y$ in (1),
$$\therefore \quad \tfrac{1}{25}y^2+\tfrac{1}{5}y^2=6,$$
$$\therefore \quad y^2=25,$$
$$\therefore \quad y=\pm 5,$$
$$\therefore \quad x=\mp 1.$$

Hence the four solutions are :
$$\begin{matrix} x=3, \\ y=1. \end{matrix} \quad \text{or} \quad \begin{matrix} x=-3, \\ y=-1. \end{matrix} \quad \text{or} \quad \begin{matrix} x=-1, \\ y=5. \end{matrix} \quad \text{or} \quad \begin{matrix} x=1, \\ y=-5. \end{matrix}$$

Verify each of these four pairs of roots.

If we had grouped (3) with (2), the results would have been the same. Show that this is true.

In this type, terms of the first degree were absent from both equations. The expression on the left in each is homogeneous, that is, every term is of the same degree. For this reason, this is called a **homogeneous system**.

The pupil should be on the look out for special methods of obtaining from the given equations an equation of the first degree. Here we might have done so by simply adding the equations and taking the square root. Solve it by this method.

EXERCISE 183

Solve and verify 1–9:

1. $3x^2-5y^2=28,\quad 3xy-4y^2=8.$

2. $2x^2-3y^2=23,\quad 2xy-3y^2=3.$

3. $x^2-xy+y^2=21,\quad 2xy-y^2=15.$

4. $2x^2-3xy=14,\quad 3y^2-x^2+1=0.$

5. $x^2+xy=66,\quad x^2-y^2=11.$

6. $x^2-xy=54,\quad xy-y^2=18.$

7. $x^2+2xy=32,\quad 2y^2+xy=16.$

8. $3x^2-5xy+2y^2=14,\quad 2x^2-5xy+3y^2=6.$

9. $x^2-4y^2=20,\quad xy=12.$

10. $x^2-3y^2=4,\quad x^2+xy+y^2=28.$

11. $x^2+xy+y^2=7,\quad 3x^2-1=xy.$

12. $32y^2=2xy+11,\quad x^2+4y^2=10.$

13. $2x^2-9xy+9y^2=5,\quad 4x^2-10xy+11y^2=35.$

14. $x^2+xy+y^2=7,\quad 2x^2+3xy+4y^2=24.$

15. $3x^2-3xy+2y^2=2x,\ 2x^2+3y^2-xy=4x.$

16. Find, to two decimals, the real values of x and y which satisfy $x^2-xy=20$ and $3xy-y^2=50$.

17. When a number is multiplied by the digit on the left the product is 105; when the sum of the digits is multiplied by the digit on the right the product is 40. Find the number.

198. Special Methods.

Since $(x+y)^2=(x-y)^2+4xy$, it follows that if the values of any two of the quantities $x+y$, $x-y$ and xy are given, the remaining one can be found.

Ex. 1.—Solve

$$x+y=11, \tag{1}$$

$$xy=18. \tag{2}$$

Squaring (1), $x^2+2xy+y^2=121.$

From (2), $4xy=72.$

Subtracting, $x^2-2xy+y^2=49,$

$$\therefore\ x-y=\pm 7.$$

If $x+y=11$,
and $x-y=7$.

$x=9$, $y=2$.

If $x+y=11$,
and $x-y=-7$.

$x=2$, $y=9$.

Hence there are two solutions :

$$x=9 \text{ or } 2,$$
$$y=2 \text{ or } 9.$$

Ex. 2.—Solve $\qquad x-y=11, \ xy=60.$

Find $(x+y)^2$ by adding $4xy$ to $(x-y)^2$ and complete the solution.

Ex. 3.—Solve

$$x^3+y^3=35, \qquad (1)$$
$$x+y=5. \qquad (2)$$

Dividing (1) by (2), $\quad x^2-xy+y^2=7, \qquad (3)$

Squaring (2), $\quad x^2+2xy+y^2=25.$

Subtracting, $\quad 3xy=18,$

$$\therefore \quad xy=6. \qquad (4)$$

Subtracting (4) from (3), $\quad x^2-2xy+y^2=1,$

$$\therefore \quad x-y=\pm 1.$$

Complete the solution as before.

Also solve by substituting $x=5-y$ from (2) in (1).

Ex. 4.—Solve

$$x^4+x^2y^2+y^4=91, \qquad (1)$$
$$x^2+xy+y^2=13. \qquad (2)$$
$$x^4+x^2y^2+y^4=(x^2+xy+y^2)(x^2-xy+y^2).$$

Dividing (1) by (2), $\quad x^2-xy+y^2=7. \qquad (3)$

Subtracting (3) from (2), $\quad 2xy=6,$

$$\therefore \quad xy=3. \qquad (4)$$

Adding (2) and (4) $\quad x^2+2xy+y^2=16,$

$$\therefore \quad x+y=\pm 4. \qquad (5)$$

Similarly from (3) and (4), $\quad x-y=\pm 2. \qquad (6)$

(5) and (6) can be grouped in four ways, thus :

$$\begin{matrix} x+y=4, \\ x-y=2. \end{matrix} \text{ or } \begin{matrix} x+y=-4, \\ x-y=-2. \end{matrix} \text{ or } \begin{matrix} x+y=4, \\ x-y=-2. \end{matrix} \text{ or } \begin{matrix} x+y=-4, \\ x-y=2. \end{matrix}$$

From these four solutions are obtained :

$$\begin{matrix} x=3, \ -3, \ 1, \ -1, \\ y=1, \ -1, \ 3, \ -3. \end{matrix} \text{ or } \begin{matrix} x=\pm 3 \text{ or } \pm 1, \\ y=\pm 1 \text{ or } \pm 3. \end{matrix}$$

Ex. 5.—Solve $(x+y)^2-5(x+y)-6=0,$ (1)

$xy=8.$ (2)

Factoring (1), $(x+y-6)(x+y+1)=0,$

$\therefore\ x+y=6$ or $-1.$

Now solve $x+y=6,\ xy=8.$ and $x+y=-1,\ xy=8.$

EXERCISE 124

Solve, by finding $x+y$ and $x-y$, and verify.

1. $x+y=8,\ xy=15.$

2. $x-y=4,\ xy=12.$

3. $x^2+y^2=25,\ x-y=1.$

4. $x^2+y^2=61,\ x+y=11.$

5. $(x-y)^2=1,\ xy=30.$

6. $x^2-xy+y^2=57,\ x-y=8.$

7. $x^2+xy+y^2=19,\ x+y=5.$

8. $x^2-xy+y^2=79,\ x+y=13.$

9. $5x^2+xy+5y^2=23,\ x+y=1.$

10. $x^2+y^2=89,\ xy=40.$

11. $x^2-7xy+y^2=-101,\ xy=30.$

12. $2x^2+3xy+2y^2=8,\ xy=-6.$

13. $x^3-y^3=19,\ x-y=1.$

14. $x^3+y^3=1064,\ x+y=14.$

15. $x^2-xy+y^2=39,\ x^3+y^3=351.$

16.* $x^4+x^2y^2+y^4=21,\ x^2+xy+y^2=7.$

17. $x^4+x^2y^2+y^4=133.\ x^2-xy+y^2=7.$

18. $x^4-x^2y^2+y^4=13,\ xy=2.$

19. $(x+y)^2-3(x+y)-28=0,\ x-y=3.$

20. $(x-y)^2-7(x-y)+12=0,\ xy=12.$

21. $x^2y^2-27xy+180=0,\ x+y=8.$

22. The perimeter of a rectangle is 34 inches and the diagonal is 13 inches. Find the sides.

23. The diagonal of a rectangle is 25 and the area is 300. Find the sides.

24. The sum of two numbers is 12 and the sum of their squares is 72·5. Find the numbers.

25. The product of two numbers is 270. If each number is decreased by 3 the product will be 180. Find the numbers.

26. The sum of two numbers is 10 and the sum of their reciprocals is $\frac{5}{12}$. Find the numbers.

27. Solve $(x-1)(y+2)=9,\ 2xy=15$.

28. A and B are two squares. The area of A is 63 square inches more than B, and the perimeter of A is 12 inches more than B. Find the side of each.

29. Find two numbers whose product is 1 and the sum of whose reciprocals is $2\frac{1}{12}$.

30. Solve $x^3-8y^3=56,\ x-2y=2$.

31. The sum of the two digits of a number is $\frac{1}{4}$ of the number. The sum of the squares of the digits is 4 less than the number. Find the number.

32. The area of a rectangle is 1161 square yards, and its perimeter is 140 yards. Find the dimensions.

33. Solve $\frac{1}{x}+\frac{1}{y}=\cdot 3,\ \frac{1}{x^2}-\frac{1}{y^2}=\cdot 0\dot{3}$.

34. The sum of a number of two digits and the number formed by reversing the digits is 121. The product of the digits is 28. Find the number.

35. Find the sides of a right-angled triangle whose perimeter is 24 inches and whose area is 24 square inches.

36. Prove, algebraically, that if two rectangles have equal area and equal perimeters, they are equal in all respects.

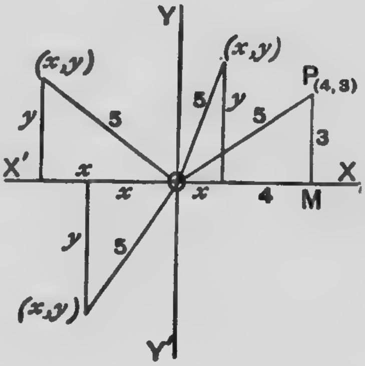
37. Solve $x^2+xy+y^2=7{\cdot}75,\ x^2-xy+y^2=5{\cdot}25$.

38. What must be the dimensions of a rectangular field containing $7\frac{1}{2}$ acres, if the greatest distance from any point in its boundary to any other point is 50 rods?

39. The sum of the radii of two circles is 8 inches and the sum of their areas is $\frac{5}{9}$ of the area of a circle whose radius is 9 inches. What are their radii ?

40. What must be the length of a rectangular field that contains a square rods and which can be enclosed by a fence b rods long.

199. Graphical Methods. What is the distance of the point $P(4, 3)$ from the origin O ?



Since $OP^2=OM^2+MP^2$,

$\therefore\ OP^2=4^2+3^2=25,$

$\therefore\ OP\ =5.$

If any point (x,y) is the same distance from the origin that P is, then the point (x,y) must lie on a circle whose radius is 5 and whose centre is O. But the square of the distance of the point (x,y) from the origin is x^2+y^2,

$$\therefore\ x^2+y^2=25.$$

It is thus seen that *the equation* $x^2+y^2=25$ *represents a circle whose radius is* 5 *and whose centre is the origin.*

Similarly, $x^2+y^2=16$, $x^2+y^2=100$, $x^2+y^2=18$, represent circles with the origin as centres and whose radii respectively are 4, 10, $\sqrt{18}$.

It is seen that it is a simple matter to draw the graph of the equation of the circle in the form $x^2+y^2=r^2$. All we require to do is to describe with the compasses a circle whose centre is the origin and whose radius is r.

When the radius is a surd as in $x^2+y^2=18$, it is simpler to find a pair of values of x and y which satisfy the equation. Here $x=3$, $y=3$ satisfies the equation, and the circle is then described through the point (3, 3).

200. Graphical Solution of Simultaneous Equations.

Solve $x^2+y^2=25,$ (1)

$x-y=1.$ (2)

(1) represents a circle whose radius is 5.

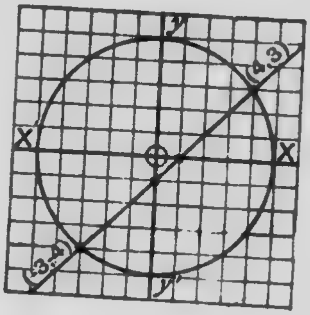
(2) represents a straight line, two points on which are (1, 0) and (0, −1).

The graphs of (1) and (2) are shown in the diagram.

The line cuts the circle at the points (4, 3) and (−3, −4).

∴ the roots of the given equations are

(4,3) and (−3, −4).

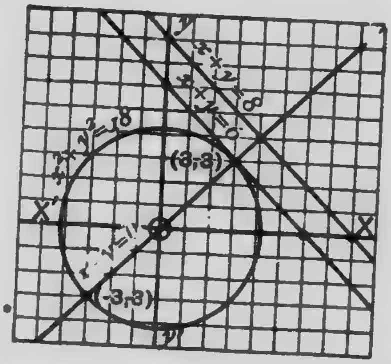


201. Equal and Imaginary Roots.

Solve, (1) $x^2+y^2=18,\ x-y=0.$

(2) $x^2+y^2=18,\ x+y=6.$

(3) $x^2+y^2=18,\ x+y=8.$



The diagram shows that the roots of (1) are

(3,3) or (−3, −3);

of (2) are

(3,3) or (3,3).

The roots of (2) are equal, as the line $x+y=6$ touches the circle at the point (3,3). We might say that in this case the line meets the circle at two points which happen to be coincident.

The diagram shows that the line $x+y=8$ does not meet the circle at all, and there are no real values of x and y which will satisfy (3). The roots in this case are imaginary.

Solve these equations by the usual methods and see if the results agree with the diagram.

EXERCISE 195

1. On the same sheet draw the graphs of the circles whose equations are $x^2+y^2=4$, $x^2+y^2=9$, $x^2+y^2=13$, $x^2+y^2=34$.

2. Solve graphically $x^2+y^2=13$, $x-y=1$.

3. Find graphically the positive integral roots of $x^2+y^2=25$ and $2x+3y=18$; $x^2+y^2=10$ and $2x-y=5$. Approximate to the other roots.

4. The sum of two numbers is 8 and the sum of their squares is 25. Show, graphically, that this is impossible. Is it impossible if the sum of the numbers is 7 instead of 8?

EXERCISE 196 (Review of Chapter XXII)

1. Solve $x+y=28$, $x^2-y^2=336$.

2. Solve $5x-2y=12$, $25x^2-4y^2=96$.

3. The sum of two numbers is 10 and the sum of their squares is 58. Find the numbers.

4. Solve $2x-3y=4$, $x^2+y^2=29$.

5. Solve $3x-4y=4$, $2x^2+3xy=56$.

6. The sum of two numbers is 5 and the sum of their reciprocals is $\frac{5}{6}$. Find the numbers.

7. Solve $x^2+xy+2y^2-2x-7y+5=0$, $x+y=3$.

8. Solve $x^2+xy-6y^2=0$, $x^2+3xy-y^2=36$.

9. A field whose length is to its breadth as 3 to 2 contains 664 square rods more than one whose length is to its breadth as 2 to 1. The difference of their perimeters is 60 rods. Find the dimensions of each field.

10. Solve $x^2+2xy=55$, $xy+2y^2=33$.

11. Solve $2x^2+3xy=8$, $y^2-2xy=20$.

12. The area of a rectangle is 300 square feet. If the length is decreased by 2 feet and the width by 3 feet, the area would be 216 square feet. Find the dimensions.

13. Solve $x(x+y)=150$, $y(x+y)=75$.

14. Solve $x(x-y)=15$, $y(x+y)=14$.

15. Sodding a lawn at 9 cents a square yard costs $108. If it had been 10 yards longer and 6 yards wider the cost would have been half as much again. Find the dimensions.

16. Solve $x^3-y^3=126$, $x^2+xy+y^2=21$.

17. Solve $x^2+3xy-5y^2+2x-y=12$, $x+y=7$.

18.* If $(x+y)^2-7(x+y)+12=0$ and $x^2y^2-6xy+8=0$, find the values of $x+y$ and xy, and thus solve these equations for x and y.

19. The product of two numbers is 28 and their difference is 5. Find the sum of their squares, without finding the numbers.

20. Solve $8x^3+y^3=280$, $2x+y=10$.

21. Solve $y=x+\sqrt{2}$, $x^2+y^2=1$.

22. Find two positive integers whose sum multiplied by the greater is 192 and whose difference multiplied by the less is 32.

23. Solve $\dfrac{a^2}{x^2}+\dfrac{b^2}{y^2}=10$, $\dfrac{ab}{xy}=3$.

24. If $12x^2-41xy+35y^2=0$, find the values of $\dfrac{x}{y}$.

25. The product of two numbers is 6 and the difference of their squares is 5. Find the numbers.

26. Solve $\dfrac{x^2}{y^2}+\dfrac{x}{y}=6$, $x-y=4$.

27. Solve $(x+y)(x+2y)=300$, $\dfrac{x}{y}+\dfrac{2y}{x}=3$.

28. A regiment consisting of 1625 men is formed into two solid squares, one of which has 15 more men on a side than the other. What is the number on a side of each?

29. Solve $\dfrac{1}{x}+\dfrac{2}{y}=8$, $\dfrac{1}{x^2}+\dfrac{4}{y^2}=40$.

30. Solve $\dfrac{1}{x}-\dfrac{1}{y}=\dfrac{1}{12}$, $\dfrac{4}{x^2}+\dfrac{6}{y^2}=\dfrac{5}{12}$.

31. The difference of two numbers is 15 and half of their product equals the cube of the less. Find the numbers.

32. Solve $x^2+3y^2=37$, $xy=10$.

33. Solve $x+\dfrac{4}{y}=4$, $y-\dfrac{3}{x}=3$.

34. Two men start to meet each other from towns which are 25 miles apart. One takes 18 minutes longer than the other to walk a mile and they meet in 5 hours. How fast does each walk?

35. Solve $\frac{1}{x}+\frac{1}{y}=\frac{1}{3}$, $\frac{1}{x^2}+\frac{9}{y^2}=\frac{1}{8}$.

36. Solve $(x+y)^2-x-y=20$, $xy=6$.

37. The difference of the cubes of two consecutive odd numbers is 218. Find the numbers.

38. Solve $x^4-x^2y^2+16y^4=28$, $x^2+3xy+4y^2=14$.

39. Solve $x^2+y=y^2+x=3$.

40. The diagonal of a rectangle is d, and the difference of the sides is s. What are the lengths of the sides? Apply the formula thus obtained to find the sides of a rectangle whose diagonal is 13 inches, and one side is 7 inches longer than the other.

41. Solve $9x^2+y^2-21(3x+y)+128=0$, $xy=4$. (Make the first equation a quadratic in $3x+y$, by adding to $9x^2+y^2$ what is necessary to make a complete square.)

42. Solve $x^2+4y^2-18x-36y+112=0$, $xy=8$

43. Solve $x^3+y^3=126$, $x^2y+xy^2=30$.

CHAPTER XXIII

INDICES

EXERCISE 187 (Oral)

1. What are the values of 3^2, 2^3, 1^4, 1^{10}, 0^2 ?

2. Simplify 3×2^2; 3×10^3; 5×0^2; $0^2\div 4$.

3. When $x=10$, what are the values of:

$$x^2,\ 6x^2,\ 200\div x,\ 500\div x^2,\ 6x^3\div x^2\ ?$$

4. Give the values of $(-1)^2$, $(-1)^3$, $(-1)^4$, $(-1)^{37}$, $(-1)^{98}$.

5. What are the values of $(-2)^2$, $(-2)^5$, $(-2)^6$?

6. Find the difference between 2^3 and 3^2, 2^5 and 5^2.

7. What does x^4 mean ? How many factors are there in $x^3\times x^5$?

8. Express in the simplest form $a^2\times a^3\times a^4$.

9. How many factors will remain when x^7 is divided by x^5 ? What is the quotient ?

10. What are the values of: $x^4\div x^2$, $x^{10}\div x^5$.

$$\frac{x^5}{x^3},\ \frac{a^{20}}{a^{10}},\ \frac{\pi r^2}{\pi r},\ \frac{a^4b^3}{a^2b^2}\ ?$$

11. What does $(a^2)^3$ mean ? Read its value without the brackets.

12. State the value in the simplest form of:

$$(x^2)^2,\ (y^3)^2,\ (y^2)^3,\ (a^2)^4,\ (a^2)^{10}.$$

13. What does $(ab)^4$ mean ? What does $\left(\frac{a}{b}\right)^3$ mean ? Read their values without brackets.

14. Express as powers of 10: 100, 1000, 10,000, 10×100, $10^2\times 10^3$, $10^5\div 10^3$.

15. Simplify $(-1)^2\times(-1)^3\times(-1)^4$; $(-a)^3\times(-a)^4\times(-a)$

16. What is the value of x if

$$10^x=1000,\ 2^x=16,\ 5^x=125,\ 3^x=81\ ?$$

17. Express 32,794 in descending powers of 10.

202. Definitions of a^m. As a^3 is the product of three factors each equal to a, so a^m is the product of m factors each equal to a.

$$a^m=a.a.a\ .\ .\ .\ \text{to } m \text{ factors.}$$

Here it is understood that m is a positive integer.

203. The Index Laws. We have already seen that:

(1) $a^3\times a^4=a^{3+4}=a^7.$

(2) $a^5\div a^3=a^{5-3}=a^2.$

(3) $(a^2)^3=a^{2\times 3}=a^6.$

(4) $(ab)^4=a^4b^4.$

(5) $\left(\frac{a}{b}\right)^3=\frac{a^3}{b^3}.$

Let us now express these statements in general form, using letters to denote the indices.

(1) $a^m\times a^n=a^{m+n}.$

(2) $a^m\div a^n=a^{m-n}.$

(3) $(a^m)^n=a^{mn}.$

(4) $(ab)^m=a^mb^m.$

(5) $\left(\frac{a}{b}\right)^m=\frac{a^m}{b^m}.$

These are called the **index laws**. The letters m and n represent any positive integers, and in (2) $m>n$ (m is greater than n), to make the division possible. The laws, as stated in the general form, may be proved as in particular cases.

204. Law I. Law for Multiplication. $a^m \times a^n = a^{m+n}$.

By definition,

$$a^m = a\,.\,a\,.\,a\,\ldots \text{ to } m \text{ factors}.$$

$$a^n = a\,.\,a\,.\,a\,\ldots \text{ to } n \text{ factors},$$

$$\begin{aligned} \therefore\quad a^m \times a^n &= (a\,.\,a\,.\,a\,\ldots \text{ to } m \text{ factors})(a\,.\,a\,.\,a\,\ldots \text{ to } n \text{ factors}). \\ &= a\,.\,a\,.\,a\,\ldots \text{ to } (m+n) \text{ factors}, \\ &= a^{m+n}, \text{ by definition.} \end{aligned}$$

Also, $a^m \times a^n \times a^p = a^{m+n} \times a^p,$

$$= a^{m+n+p}.$$

205. Law II. Law for Division. $a^m \div a^n = a^{m-n}$.

$$\begin{aligned} \frac{a^m}{a^n} &= \frac{a\,.\,a\,.\,a\,\ldots \text{ to } m \text{ factors}}{a\,.\,a\,.\,a\,\ldots \text{ to } n \text{ factors}}, \\ &= a\,.\,a\,.\,a\,\ldots \text{ to } (m-n) \text{ factors, if } m>n, \\ &= a^{m-n}. \end{aligned}$$

Here the n factors in the denominator cancel with an equal number in the numerator, leaving $m-n$ factors in the numerator.

If, however, $n>m$, the n factors in the numerator cancel with an equal number in the denominator, leaving $n-m$ factors in the denominator.

$\therefore$ when $m>n$, $\qquad a^m \div a^n = a^{m-n},$

and when $n>m$, $\qquad a^m \div a^n = \dfrac{1}{a^{n-m}}.$

206. Law III. Law of Powers. $(a^m)^n = a^{mn}$.

$$\begin{aligned} (a^m)^n &= a^m\,.\,a^m\,.\,a^m\,\ldots \text{ to } n \text{ factors}, \\ &= (a\,.\,a\,\ldots \text{ to } m \text{ factors})(a\,.\,a\,\ldots \text{ to } m \text{ factors})\ldots \text{ the brackets being repeated } n \text{ times}, \\ &= a\,.\,a\,.\,a\,\ldots \text{ to } mn \text{ factors}, \\ &= a^{mn}. \end{aligned}$$

Also, $\{(a^m)^n\}^p = (a^{mn})^p = a^{mnp}$.

207. Law IV. Power of a Product. $(ab)^n = a^n b^n$.

$$\begin{aligned} (ab)^n &= ab\,.\,ab\,.\,ab\,\ldots \text{ to } n \text{ pairs of factors}, \\ &= (a\,.\,a\,.\,a\,\ldots \text{ to } n \text{ factors})(b\,.\,b\,.\,b\,\ldots \text{ to } n \text{ factors}). \\ &= a^n b^n. \end{aligned}$$

Also, $(abc)^n = (ab)^n\,.\,c^n = a^n b^n c^n$.

208. Law V. Power of a Quotient. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.

$$\left(\frac{a}{b}\right)^n = \frac{a}{b}\cdot\frac{a}{b}\cdot\frac{a}{b}\ldots \text{ to } n \text{ factors,}$$
$$= \frac{a\,.\,a\,.\,a\ldots \text{ to } n \text{ factors}}{b\,.\,b\,.\,b\ldots \text{ to } n \text{ factors}},$$
$$= \frac{a^n}{b^n}.$$

209. We have given five index laws. They are not all independent. The second and third laws may easily be deduced from the first.

(1) When $m>n$, $a^m = a^{m-n}\times a^n$ by Law I.

$\therefore\ a^m \div a^n = a^{m-n}$, which is Law II.

(2) $a^m \times a^m = a^{m+m} = a^{2m}$, by Law I.

Similarly, $a^m \times a^m \times a^m = a^{m+m+m} = a^{3m}$,

and $a^m\,.\,a^m\,.\,a^m \ldots$ to n factors $= a^{m+m+m\ldots \text{ to } n \text{ terms}} = a^{mn}$,

$\therefore\ (a^m)^n = a^{mn}$, which is Law III.

For this reason the first law is frequently called the **fundamental index law.**

EXERCISE 188 (1-18, Oral)

Simplify :

1. $a^2\times a^3\times a^5$.
2. $x^6\times x^3\div x^7$.
3. $(x^2)^4\div x^3$.
4. $(a^2b^3)^2$.
5. $(3^2)^2$.
6. $(3^3)^2\div(3^2)^3$.
7. $(ab)^3\div a^2b$.
8. $5^6\div 5^4$.
9. $((-2)^2)^3$.
10. $\frac{6^7}{6^5}$.
11. $\frac{(-1)^7}{(-1)^4}$.
12. $\frac{(ab)^5}{(a^2b^2)^3}$.
13. $x^a\times x^b\times x^c$.
14. $a^x\,.\,a^y\,.\,a^{x-y}$.
15. $x^{m-n}\times x^{2m+n}$.
16. $x^{a+b}\div x^{a-b}$.
17. $(a^2b^3c^4)^2$.
18. $x^{a+b}\,.\,x^{b+c}\,.\,x^{c+a}$.
19.* $\left(\frac{a}{b}\right)^3\times\left(\frac{b}{c}\right)^4\times\left(\frac{c}{a}\right)^5$.
20. $x^{2a+b}\times x^{2b+c}\times x^{2a+b-c}$

21. $\left(\frac{a}{b}\right)^m \cdot \left(\frac{b}{c}\right)^m \cdot \left(\frac{c}{a}\right)^m.$

22. $\frac{(x^a)^3}{x^{b+c}} \times \frac{(x^b)^3}{x^{c+a}} \times \frac{(x^c)^3}{x^{a+b}}.$

23. $\frac{a^{m+n} \times a^{n+p}}{a^{m+n+p}}.$

24. $x^{a+b} \cdot x^{b+c} \cdot x^{c+a} \div (x^a \cdot x^b \cdot x^c)^3.$

25 Express 4^n as a power of 2 and 9^6 as a power of 3.

26. Divide 27^5 by 9^7 by expressing each as a power of 3.

27. Simplify $\frac{2^n \times 2^{n-1} \times 2^3}{4^n}$ and $\frac{9^n \times 3^{n+3}}{27^{n+1}}.$

28. Solve

$$5^{3x+1}=5^{x+3};\ 4^x=2^{x+7};\ 9^{3x+3}=27^{x+5};\ 2^x \cdot 4^x \cdot 8^x=16^{3x-3}.$$

210. Fractional, Zero and Negative Indices. We have defined a^m to mean the product of m factors each equal to a. This definition requires m to be a **positive whole number.**

Thus, the definition will tell us what a^6 means, but will not tell us what $a^{\frac{2}{3}}$ means, nor what a^{-4} means, nor what a^0 means.

If we wish to use in algebra such quantities as $a^{\frac{2}{3}}$, a^{-4}, a^0, it is necessary that we define their meanings.

Now it would be very inconvenient if we gave to these new forms of indices such meanings that the index laws, already established for positive integral indices, would not apply to them. We will, therefore, give to fractional, zero and negative indices such meanings as will make the index laws valid for them as well as for positive integral indices.

211. Meaning of a Fractional Index.

Since $x^m \times x^n = x^{m+n}$, then if we suppose that the same law applies to fractional indices, it follows that

$$x^{\frac{1}{2}} \times x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x^1 = x.$$

Thus, $x^{\frac{1}{2}}$ when multiplied by $x^{\frac{1}{2}}$ gives the product x, or the square of $x^{\frac{1}{2}}$ is x.

But we have already represented the quantity whose square is x by $\sqrt{x}$,

$$\therefore\ x^{\frac{1}{2}} = \sqrt{x}.$$

That this is a reasonable value to attach to $x^{\frac{1}{2}}$ might appear as follows:

We know that $x^4=\sqrt{x^8}$, $x^2=\sqrt{x^4}$, $x^1=\sqrt{x^2}$, the index of the quantity under the root sign in each case being half of the index in the preceding case. If now we take half of the index on each side again, it would seem but natural that $x^{\frac{1}{2}}$ should be equal to $\sqrt{x}$.

Similarly, $x^{\frac{1}{3}}\times x^{\frac{1}{3}}\times x^{\frac{1}{3}}=x^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}=x$.

$\therefore\ x^{\frac{1}{3}}=\sqrt[3]{x}$ (the cube root of x).

Also, $x^{\frac{1}{4}}=\sqrt[4]{x}$ (the fourth root of x),

and $x^{\frac{1}{n}}=\sqrt[n]{x}$ (the n^{th} root of x), where n is a positive integer

Thus, $4^{\frac{1}{2}}=\sqrt{4}=2$, $125^{\frac{1}{3}}=\sqrt[3]{125}=5$, $32^{\frac{1}{5}}=\sqrt[5]{32}=2$.

By Law III, $(x^{\frac{2}{3}})^3=x^2$,

$\therefore\ x^{\frac{2}{3}}=\sqrt[3]{x^2}$.

Similarly, $\left(x^{\frac{p}{q}}\right)^q=x^p$,

$\therefore\ x^{\frac{p}{q}}=\sqrt[q]{x^p}$, where p and q are positive integers.

We thus see that if the same laws apply to fractional indices as to positive integral indices we are led to the conclusion that $x^{\frac{p}{q}}=\sqrt[q]{x^p}$, when p and q are positive integers, that is, *when the index is a fraction, the denominator of the fraction indicates the root to be taken and the numerator the power.*

By Law III, $x^{\frac{p}{q}}=\left(x^{\frac{1}{q}}\right)^p=(x^p)^{\frac{1}{q}}$,

$\therefore\ x^{\frac{p}{q}}=(\sqrt[q]{x})^p=\sqrt[q]{x^p}$.

So that $x^{\frac{p}{q}}$ means that the p^{th} power of the q^{th} root is to be taken, or the q^{th} root of the p^{th} power.

Thus, $8^{\frac{2}{3}}=\sqrt[3]{8^2}=\sqrt[3]{64}=4$,

or $8^{\frac{2}{3}}=(\sqrt[3]{8})^2=2^2=4$.

It will be seen that it is simpler to take the root first and then the power.

Thus, $$32^{\frac{3}{5}} = (\sqrt[5]{32})^3 = 2^3 = 8.$$

212. Meaning of a Zero Index.

By Law I, $$a^0 \times a^m = a^{0+m} = a^m,$$
$$\therefore \quad a^0 = a^m \div a^m = 1.$$

Therefore, if the same law applies to zero indices as to positive integral indices, we are led to the conclusion that *any quantity (zero excepted) to the index zero is equal to unity.*

Thus, $$3^0 = 1,\ (5x)^0 = 1,\ (-2)^0 = 1,\ (-\tfrac{1}{2}ab)^0 = 1.$$

213. Meaning of a Negative Index.

By Law I, $$a^{-2} \times a^2 = a^{-2+2} = a^0 = 1,$$
$$\therefore \quad a^{-2} = \frac{1}{a^2}.$$

Similarly, $$a^{-p} \times a^p = a^{-p+p} = a^0 = 1,$$
$$\therefore \quad a^{-p} = \frac{1}{a^p}.$$

We thus see that, *any quantity to a negative index is equal to unity divided by the same quantity to the corresponding positive index.*

Thus, $$4^{-2} = \frac{1}{4^2} = \frac{1}{16}.$$
$$27^{-\frac{2}{3}} = \frac{1}{27^{\frac{2}{3}}} = \frac{1}{(\sqrt[3]{27})^2} = \frac{1}{9}.$$
$$(x^{\frac{3}{5}})^{-10} = x^{-6} = \frac{1}{x^6}.$$

Since $x^3 = x^4 \div x$, $x^2 = x^3 \div x$, $x^1 = x^2 \div x$, what would you naturally expect x^0 to be equal to ? What would you expect x^{-1} to be equal to ?

214. Since $a^{-p} = \dfrac{1}{a^p}$ and $a^p = \dfrac{1}{a^{-p}}$, it follows that *any factor may be removed from the numerator to the denominator of a fraction, or vice versa, by changing the sign of its index.*

Thus, $\frac{2 \,.\, 3^{-2}}{6^{-2}} = \frac{2 \,.\, 6^2}{3^2} = 8$; $\frac{x^a y^{-b}}{b^{-a}} = \frac{x^a b^a}{y^b}$; $4x^{-2}a^3 = \frac{4a^3}{x^2}$.

Ex.—Simplify $\sqrt[3]{8^2} \times \sqrt[4]{16^3}$; $\left(\frac{9a^4}{16b^{-6}}\right)^{-\frac{3}{2}}$

$$\sqrt[3]{8^2} \times \sqrt[4]{16^3} = 8^{\frac{2}{3}} \times 16^{\frac{3}{4}} = (2^3)^{\frac{2}{3}} \times (2^4)^{\frac{3}{4}} = 2^2 \times 2^3 = 32.$$

$$\left(\frac{9a^4}{16b^{-6}}\right)^{-\frac{3}{2}} = \frac{9^{-\frac{3}{2}} \,.\, a^{-6}}{16^{-\frac{3}{2}} \,.\, b^9} = \frac{16^{\frac{3}{2}}}{9^{\frac{3}{2}} a^6 b^9} = \frac{64}{27a^6b^9}.$$

EXERCISE 189 (1-32. Oral)

What is the meaning of:

1. $a^{\frac{1}{2}}$. 2. $x^{\frac{1}{3}}$. 3. $y^{\frac{1}{5}}$. 4. $a^{\frac{2}{3}}$.

5. $x^{\frac{4}{5}}$. 6. x^0. 7. a^{-2}. 8. x^{-1}.

9. x^{-4}. 10. y^{-n}. 11. $m^{-\frac{1}{2}}$ 12. $x^{-\frac{2}{3}}$.

What is the value of:

13. $9^{\frac{1}{2}}$. 14. $16^{\frac{1}{4}}$. 15. $125^{\frac{1}{3}}$. 16. $10{,}000^{\frac{1}{4}}$.

17. $4^{\frac{3}{2}}$. 18. $27^{\frac{2}{3}}$. 19. $(\frac{1}{4})^{\frac{3}{2}}$. 20. 5^{-2}.

21. 10^{-3}. 22. $4^{-\frac{1}{2}}$. 23. $8^{-\frac{1}{3}}$. 24. $(-6)^0$.

25. $8^{-\frac{2}{3}}$. 26. $(a^0)^{-2}$. 27. $(\cdot 25)^{\frac{1}{2}}$. 28. $(\cdot 16)^{\frac{3}{2}}$.

29. $(\frac{3}{4})^{-2}$. 30. $(-2)^{-4}$. 31. $3^0 \,.\, 4^0$. 32. $2^8 \,.\, 2^{-6}$.

Write with positive indices:

33. a^2b^{-3}. 34. $2a^{-3}$. 35. $\frac{a^{-2}}{b^{-3}}$. 36. $\frac{ab^{-3}}{cd^{-4}}$.

37. $\frac{a^{-2}b^{-3}}{x^{-3}y^{-4}}$. 38. $\frac{1}{x^{-5}y}$. 39. $\frac{2x^{-1}}{3y^{-2}}$. 40. $\frac{4^{-3} \,.\, x^3}{3^{-2} \,.\, y^{-3}}$.

Write without a denominator:

41. $\frac{2xy}{z^2}$. 42. $\frac{4a^3}{b^2}$. 43. $\frac{3x}{a^{-2}b^4}$. 44. $\frac{5}{a^2(c+d)^{-3}}$.

Simplify:

45.* $16^{-\frac{3}{4}}$. **46.** $\dfrac{1}{32^{-\frac{3}{5}}}$. **47.** $(\cdot04)^{-2}$. **48.** $(\cdot027)^{-\frac{2}{3}}$.

49. $25^{1\cdot5}$. **50.** $(-8)^{-\frac{2}{3}}$. **51.** $\sqrt[3]{8^{-4}}$. **52.** $16^{1\cdot25}$.

53. $\dfrac{2^{-1}}{2^{-2}-2^{-3}}$. **54.** $\left(\dfrac{50}{98}\right)^{\frac{1}{2}}$. **55.** $\left(\dfrac{9}{16}\right)^{-\frac{1}{2}}$.

56. $\dfrac{3^{-3}-2^{-3}}{3^{-1}-2^{-1}}$. **57.** $\left(\dfrac{48}{243}\right)^{\frac{3}{4}}$. **58.** $\left(\dfrac{16a^{-4}}{81b^4}\right)^{-\frac{3}{4}}$

59. Solve $x^{\frac{1}{2}}=4$, $x^{\frac{5}{3}}=32$, $x^{\frac{3}{4}}=27$, $x^{-\frac{1}{2}}=3$, $x^{-\frac{3}{2}}=8$.

215. Operations with Fractional and Negative Indices. The following examples will illustrate how the index laws may be applied to the multiplication, division, etc., of quantities involving fractional and negative indices.

The work will usually be simplified if all expressions are arranged in descending or ascending powers of some common letter.

Thus, $5+x^3+x^{-2}-x^{-1}+x^2$ would be written in descending powers of x, thus:

$$x^3+x^2+5-x^{-1}+x^{-2}.$$

Ex. 1.—Multiply $2x^{\frac{1}{2}}+3-x^{-\frac{1}{2}}$ by $3x^{\frac{1}{2}}-2-5x^{-\frac{1}{2}}$.

Ex. 2.—Divide $a-b$ by $a^{\frac{1}{3}}-b^{\frac{1}{3}}$.

(1)

$$\begin{array}{rrrrr}
2x^{\frac{1}{2}}+3 & - & x^{-\frac{1}{2}} & & \\
3x^{\frac{1}{2}}-2 & - & 5x^{-\frac{1}{2}} & & \\
\hline
6x+9x^{\frac{1}{2}} & - & 3 & & \\
-4x^{\frac{1}{2}} & - & 6 & +\ 2x^{-\frac{1}{2}} & \\
 & & -10 & -15x^{-\frac{1}{2}} & +5x^{-1} \\
\hline
6x+5x^{\frac{1}{2}} & & -19 & -13x^{-\frac{1}{2}} & +5x^{-1}
\end{array}$$

(2)

$$\begin{array}{l}
a^{\frac{1}{3}}-b^{\frac{1}{3}})\,a \qquad -b\,(a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}} \\
\qquad\qquad \underline{a-a^{\frac{2}{3}}b^{\frac{1}{3}}} \\
\qquad\qquad\qquad a^{\frac{2}{3}}b^{\frac{1}{3}} \\
\qquad\qquad\qquad \underline{a^{\frac{2}{3}}b^{\frac{1}{3}}-a^{\frac{1}{3}}b^{\frac{2}{3}}} \\
\qquad\qquad\qquad\qquad a^{\frac{1}{3}}b^{\frac{2}{3}}-b \\
\qquad\qquad\qquad\qquad \underline{a^{\frac{1}{3}}b^{\frac{2}{3}}-b}
\end{array}$$

Ex. 3.—Find the square root of $9x-12x^{\frac{1}{2}}+10-4x^{-\frac{1}{2}}+x^{-1}$.

$$
\begin{array}{r|l|l}
 & 9x-12x^{\frac{1}{2}}+10-4x^{-\frac{1}{2}}+x^{-1} & \underline{3x^{\frac{1}{2}}-2+x^{-\frac{1}{2}}} \\
 & \underline{9x} & \\
\underline{6x^{\frac{1}{2}}-2} & -12x^{\frac{1}{2}}+10 & \\
 & \underline{-12x^{\frac{1}{2}}+4} & \\
\underline{6x^{\frac{1}{2}}-4+x^{-\frac{1}{2}}} & 6-4x^{-\frac{1}{2}}+x^{-1} & \\
 & \underline{6-4x^{-\frac{1}{2}}+x^{-1}} &
\end{array}
$$

Verify by squaring $3x^{\frac{1}{2}}-2+x^{-\frac{1}{2}}$ by the method of art. 93. Also check by putting $x=1$.

EXERCISE 140

Multiply:

1.* $x^{\frac{1}{2}}+3$, $x^{\frac{1}{2}}-2$.

2. $x+x^{\frac{1}{2}}+1$, $x^{\frac{1}{2}}-1$.

3. $x^{\frac{3}{2}}-x+x^{\frac{1}{2}}-1$, $x^{\frac{1}{2}}+1$.

4. $3x-2x^{\frac{1}{2}}+5$, $x-2x^{\frac{1}{2}}$.

5. $a^{\frac{1}{2}}-1+2a^{-\frac{1}{2}}$, $a^{\frac{1}{2}}+1-2a^{-\frac{1}{2}}$.

6. $(a-a^{\frac{1}{2}}+1)^2$.

7. $x+5x^{\frac{3}{4}}+6x^{\frac{1}{2}}$, $x^{\frac{1}{4}}-1-x^{-\frac{1}{4}}$.

8. $(x^{\frac{1}{2}}+2)^4$.

9. $x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y$, $x-x^{\frac{1}{2}}y^{\frac{1}{2}}+y$.

10. $(a^{\frac{1}{2}}-1)^3$.

Divide and verify:

11. $a+5a^{\frac{1}{2}}b^{\frac{1}{2}}+6b$ by $a^{\frac{1}{2}}+2b^{\frac{1}{2}}$.

12. $x^3-x^2+x-2-2x^{-2}-2x^{-3}$ by x^2+2+2x^{-2}.

13. $x^{\frac{4}{3}}+x^{\frac{2}{3}}y^{\frac{2}{3}}+y^{\frac{4}{3}}$ by $x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$.

14. $1-5x^{\frac{1}{3}}-x$ by $1-x^{\frac{1}{3}}+3x^{\frac{2}{3}}$, as far as four terms.

Find the square root and verify:

15. $a+6a^{\frac{1}{2}}+9$ and $25x^2-10+x^{-2}$

16. $a^2+4a^{\frac{3}{2}}+6a+4a^{\frac{1}{2}}+1$.

17. $4x^{\frac{5}{3}}-20x^{\frac{4}{3}}+37x-30x^{\frac{2}{3}}+9x^{\frac{1}{3}}$.

18. $49-30x^{\frac{1}{3}}-24x^{-\frac{1}{3}}+25x^{\frac{2}{3}}+16x^{-\frac{2}{3}}$

19. Show that $\dfrac{a^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}}{a+b}=(a^{\frac{1}{3}}+b^{\frac{1}{3}})^{-1}$.

20. Divide $x^{\frac{5}{2}}-x^{-\frac{5}{2}}$ by $x^{\frac{1}{2}}-x^{-\frac{1}{2}}$.

21. Divide $10a^{3m}-32a^{m}-27a^{2m}+14$ by $2a^{m}-7$.

22. Simplify $(x+x^{\frac{1}{2}}+1)^2+(x-x^{\frac{1}{2}}+1)^2$.

23. Simplify $(\sqrt{a}+1)(\sqrt{a}-1)-(\sqrt{3a}+\sqrt{2})(\sqrt{3a}-\sqrt{2})$.

24. Find the square root of $x^2-4x\sqrt{x}+10x-12\sqrt{x}+9$.

216. **Contracted Methods.** The following examples will illustrate how contracted methods may be employed.

Ex. 1.—Multiply $x+x^{\frac{1}{2}}-4$ by $x+x^{\frac{1}{2}}+4$.

If $x+x^{\frac{1}{2}}$ be considered as a single term, the product

$$=(x+x^{\frac{1}{2}})^2-4^2, \qquad [(a+b)(a-b)=a^2-b^2.]$$
$$=x^2+2x^{\frac{3}{2}}+x-16.$$

Ex. 2.—Divide $a+b$ by $a^{\frac{1}{3}}+b^{\frac{1}{3}}$.

Since a is the cube of $a^{\frac{1}{3}}$ and b of $b^{\frac{1}{3}}$, this is similar to dividing x^3+y^3 by $x+y$.

Since
$$(x^3+y^3)\div(x+y)=x^2-xy+y^2,$$
so
$$(a+b)\div(a^{\frac{1}{3}}+b^{\frac{1}{3}})=a^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}.$$

Ex. 3.—What is the cube root of

$$8x^{\frac{3}{2}}-36xy^{\frac{1}{4}}+54x^{\frac{1}{2}}y^{\frac{1}{2}}-27y^{\frac{3}{4}}\ ?$$

This is evidently the cube of a binomial whose first term is $2x^{\frac{1}{2}}$ and last term is $-3y^{\frac{1}{4}}$.

∴ the cube root is $2x^{\frac{1}{2}}-3y^{\frac{1}{4}}$, if the given expression is a perfect cube. Check when $x=y=1$.

Using the method of art. 155, the cube root of more complicated expressions may be found.

217. Factors with Fractional or Negative Indices. If we are permitted to use fractional or negative indices, many expressions may be factored which were previously considered algebraically prime.

Ex. 1. $a-b$ may be written as the difference of two squares, thus $(a^{\frac{1}{2}})^2-(b^{\frac{1}{2}})^2$.

$$\therefore \quad a-b=(a^{\frac{1}{2}}+b^{\frac{1}{2}})(a^{\frac{1}{2}}-b^{\frac{1}{2}}).$$

Ex. 2. $3x-1-2x^{-1}$ may be factored by cross multiplication The factors are $(3x^{\frac{1}{2}}+2x^{-\frac{1}{2}})(x^{\frac{1}{2}}-x^{-\frac{1}{2}})$.

Ex. 3. x^2+xy+y^2 is an incomplete square.

It may be written $(x+y)^2-(x^{\frac{1}{2}}y^{\frac{1}{2}})^2$.

$\therefore$ the factors are $(x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y)(x-x^{\frac{1}{2}}y^{\frac{1}{2}}+y)$.

EXERCISE 141

Use contracted methods in the following:

1.* Multiply $x^{\frac{1}{2}}-2$ by $x^{\frac{1}{2}}+2$; $a^{\frac{3}{2}}-b^{\frac{3}{2}}$ by $a^{\frac{3}{2}}+b^{\frac{3}{2}}$

2. Multiply $a^{\frac{1}{2}}-1+a^{-\frac{1}{2}}$ by $a^{\frac{1}{2}}+1+a^{-\frac{1}{2}}$.

3. Find the square of $x-x^{\frac{1}{2}}-1$ and of $2a-2-a^{-1}$.

4. Write down the cube of $a^{\frac{1}{2}}+1$ and of $1-x^{\frac{1}{2}}$.

5. Multiply $x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y$, $x-x^{\frac{1}{2}}y^{\frac{1}{2}}+y$, x^2-xy+y^2.

6. Divide $x+y$ by $x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$.

7. Divide $a+2a^{\frac{1}{2}}b^{\frac{1}{2}}+b-c$ by $a^{\frac{1}{2}}+b^{\frac{1}{2}}-c^{\frac{1}{2}}$.

8. Find three factors of x^2-y^2.

9. Find a common factor of $a+a^{\frac{1}{2}}b^{\frac{1}{2}}-2b$, $a-b$.

10. Simplify $\dfrac{x+x^{\frac{1}{2}}-6}{x-5x^{\frac{1}{2}}+6}$, $\dfrac{a-b}{a^{\frac{1}{3}}-b^{\frac{1}{3}}}$, $\dfrac{a^2+ab+b^2}{a+\sqrt{ab}+b}$.

11. What is the cube root of $x^{\frac{3}{2}}-6x+12x^{\frac{1}{2}}-8$, and of

$$x^3-3x^{\frac{5}{2}}+6x^2-7x^{\frac{3}{2}}+6x-3x^{\frac{1}{2}}+1\ ?$$

12. What is the square root of $4x^{-4}+12x^{-3}+9x^{-2}$, and of

$$x^2+4x+2-4x^{-1}+x^{-2}\ ?$$

EXERCISE 142 (Review of Chapter XXIII)

1. State the index laws.

2. Explain how meanings are assigned to such quantities as $a^{\frac{1}{3}}$, a^0, a^{-2}.

3.* When $x=16$, $y=9$, find the values of:

$$(x+y)^{\frac{1}{2}},\ x^{-\frac{1}{2}}+y^{-\frac{1}{2}},\ (x^{\frac{1}{2}}+y^{\frac{1}{2}})^2.$$

4. Find the numerical values of:

$$8^{\frac{2}{3}},\ 9^{-\frac{3}{2}},\ \sqrt[3]{125^2},\ 32^{\cdot 4},\ 16^{-1\cdot 5},\ \cdot 25^{-\frac{1}{2}},\ (-64)^{-\frac{2}{3}}.$$

5. Show that $\sqrt[3]{8^2}\times\sqrt[4]{81^3}=108$.

6. Simplify $32^{-\frac{1}{5}}\div(\frac{1}{27})^{\frac{1}{3}}$ and $8\{2^{\frac{1}{3}}\div(\frac{1}{16})^{-\frac{1}{6}}\}^2$.

7. Find, to two decimals, the value of 10^x when $x=\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{3}{4}$.

8. Simplify $2^3+10^0-4^{\frac{1}{2}}-(\frac{1}{9})^{-\frac{1}{2}}+0^4+(\sqrt{\frac{1}{16}})^{-1}$.

9. Find, to three decimals, the value of $(3^{\frac{2}{3}})^{-\frac{3}{4}}\times\sqrt[4]{27}$.

10. Simplify $16^{\frac{3}{4}}+16^{\frac{1}{4}}-16^{-\frac{1}{4}}-16^{-\frac{3}{4}}$ and $32^{\frac{1}{5}}-32^{\frac{2}{5}}+32^{-\frac{1}{5}}+32^{-\frac{2}{5}}$.

11. Simplify $5^{\frac{2}{3}}\times 5^{\frac{3}{4}}\times 5^{\frac{7}{12}}\times 16^{\frac{5}{12}}\times 16^{\frac{1}{24}}\times 16^{\frac{1}{12}}$.

12. Solve $x^{\frac{3}{2}}=8$, $2^x\,.\,4^x=64$.

13. Simplify $4^{\frac{2}{3}}\times 6^{-\frac{1}{3}}\times\sqrt[3]{3}$ and $(8^{\frac{2}{3}}+4^{\frac{1}{2}})\times 16^{-\frac{3}{4}}$.

14. Simplify $\dfrac{2^{n+1}}{5^{n+1}}\times\dfrac{6^{-n}}{15^{-n}}$ and $\dfrac{3\,.\,2^n-4\,.\,2^{n-1}}{5\,.\,2^n-3\,.\,2^{n-1}}$.

15. Reduce to lowest terms:

$$\frac{a+8\sqrt{a}+15}{a+7\sqrt{a}+12},\quad \frac{3x^{\frac{1}{2}}+5x^{\frac{1}{4}}+2}{x^{\frac{1}{4}}+1},\quad \frac{a^{\frac{3}{2}}+ab}{ab-b^3}.$$

X

16. Multiply $x^{\frac{3}{2}}y^{\frac{1}{2}}-2xy+4x^{\frac{1}{2}}y^{\frac{3}{2}}$ by $x^{\frac{1}{2}}+2y^{\frac{1}{2}}$.

17. Multiply $x^{\frac{1}{2}}y^{-\frac{1}{2}}+1+x^{-\frac{1}{2}}y^{\frac{1}{2}}$ by $x^{\frac{1}{2}}y^{-\frac{1}{2}}-1+x^{-\frac{1}{2}}y^{\frac{1}{2}}$.

18. Divide x^2-y^{-2} by $x^{\frac{4}{3}}+x^{\frac{2}{3}}y^{-\frac{2}{3}}+y^{-\frac{4}{3}}$

and $a^{\frac{2}{3}}+1+a^{-\frac{2}{3}}$ by $a^{\frac{1}{3}}+1+a^{-\frac{1}{3}}$.

19. Divide $a^{4m}-b^{4m}$ by a^m-b^m.

20. The dividend is $y^{\frac{8}{3}}+2y^2-3y-2$, the quotient is $y^{\frac{2}{3}}-y^{\frac{1}{3}}-1$, the remainder is $3y^{\frac{1}{3}}-1$. Find the divisor.

21. Find the square root of $(x+x^{-1})^2-4(x-x^{-1})$.

22. What is the cube root of $\frac{1}{8}a^{3x}-\frac{1}{4}a^{2x}b^y+\frac{1}{6}a^xb^{2y}-\frac{1}{27}b^{3y}$?

23. If $x=a^2+1$ and $y=a^{-2}+1$, show that $\frac{xy+x-y}{xy-x+y}=a^2$.

24. Simplify $\cdot 008^{\frac{1}{3}}$, $1\cdot 728^{\frac{2}{3}}$, $2\cdot 25^{1\cdot 5}$, $\cdot 0625^{-\frac{3}{4}}$.

25. If $x+y=a^{\frac{1}{2}}$ and $x-y=a^{-\frac{1}{2}}$, find the values of xy and x^2+y^2 in terms of a.

26. If $2a=2^x+2^{-x}$ and $2b=2^x-2^{-x}$, find a^2-b^2.

27. Find the square root of $(e^x-e^{-x})^2+4$ and of

$$x^3-4x^{\frac{5}{2}}y^{\frac{1}{2}}+10x^2y-14x^{\frac{3}{2}}y^{\frac{3}{2}}+13xy^2-6x^{\frac{1}{2}}y^{\frac{5}{2}}+y^3.$$

28. Simplify $\frac{a^{\frac{3}{4}}b^{\frac{1}{2}}}{c^{\frac{1}{3}}}\times\frac{a^{-\frac{1}{4}}c^{-\frac{1}{3}}}{b^{-\frac{1}{3}}}\div\frac{a^{-\frac{1}{2}}b^{-\frac{1}{6}}c^{-\frac{2}{3}}}{c}$.

29. Factor x^2-y, $x-5x^{\frac{1}{2}}-6$, $x-1$, $4a-b^2$ and

$$x^2-4x+10-12x^{-1}+9x^{-2}.$$

30. If $10^{\cdot 30103}=2$, find the value of $10^{\cdot 60206}$ and $10^{1\cdot 50515}$.

31. If $7^{2\cdot 0104}=50$ and $7^{2\cdot 0593}=55$, find the value of $7^{4\cdot 0697}$.

32. Simplify $\frac{2^{n+1}}{(2^n)^{n-1}}\times\frac{(2^{n-1})^{n+1}}{4^{n+1}}$.

33. Solve $3^{x+1}+2^y=35$, $3^x+2^{y+2}=41$.

34. Divide $x-2(x^{\frac{2}{3}}-x^{-\frac{2}{3}})+2(x^{\frac{1}{3}}-x^{-\frac{1}{3}})-x^{-1}$ by $x^{\frac{1}{3}}-x^{-\frac{1}{3}}$.

35. Show that $x^{x\sqrt{x}}=(x\sqrt{x})^x$ is satisfied by $x=2\frac{1}{4}$.

36. Find the square root of

$$\tfrac{1}{4}(2\sqrt{x})^4-2x^{\frac{1}{4}}+x+4x^3+\sqrt{x^{-1}}-\frac{4x^2}{\sqrt[4]{x^3}}.$$

CHAPTER XXIV

SURDS AND SURD EQUATIONS

218. In Chapter XVIII. we have already dealt with elementary quadratic surds It was there shown by squaring that $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$.

We might now deduce it from the index laws.

From Law IV, $\qquad (ab)^n = a^n b^n$.

Let $n = \frac{1}{2}$, $\qquad \therefore\ (ab)^{\frac{1}{2}} = a^{\frac{1}{2}} b^{\frac{1}{2}}$,

$$\therefore\ \sqrt{ab} = \sqrt{a} \times \sqrt{b}.$$

Similarly, $\qquad (ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}$,

$$\therefore\ \sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}.$$

219. Orders of Surds. We have already defined a quadratic surd as one in which the square root is to be taken. A cubic surd is one in which the cube root is to be taken. When higher roots are to be taken as the fourth, fifth . . . n^{th}, they are called surds of the fourth, fifth . . . n^{th} orders.

220. Changing the Order of a Surd. A surd of any order may be expressed as an equivalent surd of any order which is a multiple of the given order.

Thus, $\qquad \sqrt{x} = x^{\frac{1}{2}} = x^{\frac{2}{4}} = x^{\frac{3}{6}} = x^{\frac{n}{2n}}$,

$$\therefore\ \sqrt{x} = \sqrt[4]{x^2} = \sqrt[6]{x^3} = \sqrt[2n]{x^n}.$$

Similarly, $\qquad x^{\frac{1}{3}} = x^{\frac{2}{6}} = x^{\frac{3}{9}} = x^{\frac{n}{3n}}$,

$$\therefore\ \sqrt[3]{x} = \sqrt[6]{x^2} = \sqrt[9]{x^3} = \sqrt[3n]{x^n}.$$

221. To Compare Surds of Different Orders. Any two surds may be reduced to surds of the same order and their values compared.

Thus, to compare the values of $\sqrt{2}$ and $\sqrt[3]{3}$,

$$\sqrt{2}=2^{\frac{1}{2}}=2^{\frac{3}{6}}=\sqrt[6]{2^3}=\sqrt[6]{8}.$$

$$\sqrt[3]{3}=3^{\frac{1}{3}}=3^{\frac{2}{6}}=\sqrt[6]{3^2}=\sqrt[6]{9}.$$

It is thus seen that $\sqrt[3]{3}$ is greater than $\sqrt{2}$.

222. Changes in the Form of Surds. Any mixed surd can be expressed as an entire surd.

Thus,

$$2\sqrt[3]{5}=\sqrt[3]{2^3}\times\sqrt[3]{5}=\sqrt[3]{40}.$$

$$a\sqrt[n]{b}=\sqrt[n]{a^n}\times\sqrt[n]{b}=\sqrt[n]{a^nb}.$$

$$(m+n)\sqrt{\frac{m-n}{m+n}}=\sqrt{(m+n)^2\cdot\frac{m-n}{m+n}}=\sqrt{m^2-n^2}.$$

Conversely,

$$\sqrt[3]{250}=\sqrt[3]{125}\times\sqrt[3]{2}=5\sqrt[3]{2}.$$

$$\sqrt[n]{ab^{2n}}=\sqrt[n]{b^{2n}}\times\sqrt[n]{a}=b^2\sqrt[n]{a}.$$

$$\sqrt[3]{\tfrac{4}{5}}=\sqrt[3]{\tfrac{100}{125}}=\sqrt[3]{\tfrac{1}{125}}\times\sqrt[3]{100}=\tfrac{1}{5}\sqrt[3]{100}.$$

$$\sqrt[3]{-81}=\sqrt[3]{-27}\times\sqrt[3]{3}=-3\sqrt[3]{3}.$$

EXERCISE 148

Express as mixed surds :

1. $\sqrt{27}$, $\sqrt{100a}$ $\sqrt{5b^2}$, $\sqrt{8a^2b}$, $\sqrt{32x^3y}$, $\sqrt{363a^4b^3}$.
2. $\sqrt[3]{16}$, $\sqrt[3]{8a^4}$, $\sqrt[3]{54x^4}$, $\sqrt[3]{125a^2b}$, $\sqrt[3]{-81a^5}$, $\sqrt[3]{-\frac{1}{8}a^4}$.
3. $\sqrt[4]{32}$, $\sqrt[4]{243}$, $\sqrt[4]{\frac{1}{16}a^5}$, $\sqrt[5]{64}$, $\sqrt{8x^2+16xy+8y^2}$.

Express as entire surds :

4. $2\sqrt{3}$, $10\sqrt{2}$, $3\sqrt{a}$, $a\sqrt{5}$, $ab\sqrt{b}$, $(a-b)\sqrt{a-b}$.
5. $2\sqrt[3]{3}$, $3\sqrt[3]{7}$, $-\frac{1}{2}\sqrt[3]{16}$, $a\sqrt[3]{a^2b}$, $\frac{1}{x}\sqrt[3]{x^4y}$, $2\sqrt[4]{5}$.
6. $(a+b)\sqrt{\frac{a-b}{a+b}}$, $(m+n)\sqrt{\frac{1}{m^2-n^2}}$, $\frac{x-y}{x+y}\sqrt{\frac{x^2+xy}{xy-y^2}}$.

7.* Reduce $\sqrt[3]{2}$, $\sqrt{3}$ to surds of the same order. Also reduce

$\sqrt[3]{2}$ and $\sqrt[4]{3}$; $\sqrt{2}$, $\sqrt[3]{3}$ and $\sqrt[4]{5}$.

8. Which is the greater:

$3\sqrt{2}$ or $2\sqrt{3}$; $5\sqrt{6}$ or $7\sqrt{3}$; $\sqrt{5}$ or $\sqrt[3]{10}$; 1·26 or $\sqrt[3]{2}$; $\sqrt[3]{3}$ or $\sqrt[4]{5}$?

Reduce to like surds and simplify:

9. $\sqrt{8}+\sqrt{18}+\sqrt{98}$.
10. $\sqrt{500}+\sqrt{80}-\sqrt{20}$.
11. $3\sqrt{32}+5\sqrt{50}-\frac{1}{2}\sqrt{128}$.
12. $\sqrt[3]{16}-\sqrt[3]{128}+\sqrt[3]{250}$.
13. $\sqrt[3]{96}-2\sqrt[3]{-12}+\sqrt[3]{324}$.
14. $\sqrt[4]{32}+\sqrt[4]{162}+\sqrt[4]{1250}$.
15. $\sqrt{75}-3\sqrt{12}+5\sqrt{300}+2\sqrt{48}-7\sqrt{147}+3\sqrt{\frac{1}{3}}$.
16. $x\sqrt{x+y}+\sqrt{x^3+x^2y}-\sqrt{(x+y)^3}-\sqrt{(x^2-y^2)(x-y)}$.
17. Express as equivalent surds of a lower order:

$\sqrt[4]{9}$, $\sqrt[6]{125}$, $\sqrt[4]{x^4y^2}$, $\sqrt[6]{16x^2y^6}$, $\sqrt[10]{32}$.

18. If $\sqrt[3]{2}=1{\cdot}26$ approximately, find the values of:

$\sqrt[3]{16}$, $\sqrt[3]{54}$, $\sqrt[3]{2000}$, $\sqrt[3]{\frac{1}{4}}$, $\sqrt[3]{{\cdot}002}$, $\frac{1}{3}\sqrt[3]{6{\cdot}75}$.

19. Show that $2\times\sqrt{2}\times\sqrt[4]{2}\times\sqrt[8]{4}=4$.

223. Rationalizing a Surd Denominator. When the numerical value of a fraction with a surd denominator is required, the value is more easily obtained when the denominator is rational (art. 165).

When the denominator contains only two terms, it may be rationalized by multiplying by its conjugate (art. 164).

EXERCISE 144

Multiply:

1. $2\sqrt{3}$, $3\sqrt{5}$.
2. $\sqrt{2ax}$, $\sqrt{3ax}$.
3. $\sqrt{x}$, $\sqrt{xy}$.
4. $\sqrt[3]{4}$, $\sqrt[3]{5}$.
5. $6\sqrt{14}$, $\frac{1}{2}\sqrt{21}$.
6. $\sqrt[3]{-2}$, $\sqrt[3]{-4}$.
7. $\sqrt[3]{a-b}$, $\sqrt[3]{a+b}$, $\sqrt[3]{a^2+b^2}$.
8. $\sqrt{x}+2$, $\sqrt{x}-3$, $\sqrt{x}-2$, $\sqrt{x}+3$.

9. $\sqrt{2}+\sqrt{3}-\sqrt{5}$, $\sqrt{2}+\sqrt{3}+\sqrt{5}$.

10. $\sqrt[3]{a}-1$, $\sqrt[3]{a}-2$, $\sqrt[3]{a}+3$.

11. $\sqrt{6-\sqrt{11}}$, $\sqrt{6+\sqrt{11}}$.

12. $(\sqrt{18}+\sqrt{12}+\sqrt{8})^2$.

What is the simplest quantity by which the following must be multiplied, to produce rational products? What is the product in each case?

13. $3\sqrt{2}$.
14. $2\sqrt{5}$.
15. $\sqrt{32}$.
16. $\sqrt{64}$.
17. $\sqrt{512}$.
18. $\sqrt[3]{2}$.
19. $\sqrt[3]{-3}$.
20. $\sqrt[4]{48}$.
21. $3-\sqrt{2}$.
22. $\sqrt{a}+b$.
23. $3\sqrt{2}-2\sqrt{3}$.
24. $a\sqrt{b}-b\sqrt{a}$.

Rationalize the denominator of:

25.* $\dfrac{15\sqrt{2}}{\sqrt{5}}$.
26. $\sqrt{\tfrac{1}{3}}$.
27. $\sqrt{1{\cdot}8}$.
28. $\dfrac{4+2\sqrt{2}}{2+2\sqrt{2}}$.
29. $(\sqrt{8}+\sqrt{3})^{-1}$.
30. $\dfrac{a^2}{\sqrt{a^2+b^2}-b}$.
31. $\dfrac{a+b-c}{\sqrt{a+b}+\sqrt{c}}$.
32. $\dfrac{\sqrt{x+y}-\sqrt{x-y}}{\sqrt{x+y}+\sqrt{x-y}}$.

33. Find, to three decimal places, the value of
$22\div(3\sqrt{2}-\sqrt{7})(2\sqrt{2}+\sqrt{7})$ and of $(5+\sqrt{7})\div(3+\sqrt{7})$.

34. If $x=\dfrac{2-\sqrt{3}}{2+\sqrt{3}}$ and $y=\dfrac{2+\sqrt{3}}{2-\sqrt{3}}$, find the value of x^2+y^2.

35. Simplify $(2\sqrt{3}-\sqrt{2})^2-(\sqrt{3}-\sqrt{2})^2$.

36. Simplify $\dfrac{3-\sqrt{5}}{3+\sqrt{5}}+\dfrac{4+\sqrt{5}}{4-\sqrt{5}}$ and $\dfrac{5+\sqrt{5}}{1+\sqrt{5}}+\dfrac{5+2\sqrt{5}}{2+\sqrt{5}}$.

37. Simplify $\dfrac{7-2\sqrt{5}}{4-\sqrt{5}}-\dfrac{15+6\sqrt{5}}{2+\sqrt{5}}$.

38. Show that $3-\sqrt{7}$ is a root of the equation $x^3-5x^2-4x+2=0$.

39. The three dimensions of a room are equal. If the longest diagonal from the ceiling to the floor is 18 feet, find the length of the room to the nearest inch.

224. Surd Equations. When an equation contains a single quadratic surd, and the equation is written with the surd alone on one side, the surd may be removed by squaring both sides of the equation (art. 166).

If the equation contains three terms, two or three of which are surds, the operation of squaring must be performed twice.

Ex. 1.—Solve $1+\sqrt{x}=\sqrt{x+25}$.

Squaring, $1+2\sqrt{x}+x=x+25$,

$\therefore \quad 2\sqrt{x}=24$,

$\therefore \quad \sqrt{x}=12$,

$\therefore \quad x=144$.

Verification: $1+\sqrt{x}=1+\sqrt{144}=13$.

$\sqrt{x+25}=\sqrt{169}=13$.

Solve by squaring in the form $\sqrt{x}=\sqrt{x+25}-1$ and in the form $1=\sqrt{x+25}-\sqrt{x}$, and compare the three solutions.

Ex. 2.—Solve $1-\sqrt{x}=\sqrt{x+25}$.

Squaring, $1-2\sqrt{x}+x=x+25$,

$\therefore \quad -2\sqrt{x}=24$,

$\therefore \quad -\sqrt{x}=12$,

$\therefore \quad x=144$.

Compare, line by line, this solution with Ex. 1. The answer is the same to both, although the equations are different. We have verified Ex. 1, and we know that $x=144$ is the correct result.

Let us now verify Ex. 2.

Verification: $1-\sqrt{x}=1-\sqrt{144}=-11$.

$\sqrt{x+25}=\sqrt{169}=13$.

It is seen that our attempt at verification shows that $x=144$ is not the correct root of the equation in Ex. 2.

If in verifying we could say that $\sqrt{144}$ is -12, the equation would be satisfied. But this is not allowable, as the symbol $\sqrt{}$ always represents the positive square root (art. 63).

This may be explained as follows:

(1) The equation $1-\sqrt{x}=\sqrt{x+25}$ is impossible of solution, as $\sqrt{x+25}$ is a positive quantity, and therefore $1-\sqrt{x}$ must be positive, that is, x must be less than 1. But it is evident that no value of x which is less than 1 can satisfy the equation.

(2) If we square both sides of an equation, the resulting equation is not necessarily **equivalent** to the given equation.

A simple example will show that this is the case.

Let $$x=-6.$$

Squaring, $$\therefore \quad x^2=36.$$

Now the equation $x^2=36$ has two roots $+6$ and -6, and is, therefore, not equivalent to the given equation.

This is similar to the case in which both sides of an equation are *multiplied* by a factor containing the unknown.

Let $$x=2.$$

Multiply by $x-3$, $$\therefore \quad x(x-3)=2(x-3),$$
$$\therefore \quad x^2-5x+6=0.$$

The equation $x^2-5x+6=0$, which has the roots 2 and 3, is not equivalent to the given equation.

225. Extraneous Roots. Roots which are introduced into an equation by squaring or multiplying are called **extraneous** roots.

Thus, 6 in the first equation and 3 in the second equation are extraneous roots.

Refer to Ex. 4, art. 145, where reference is made to the effect of *dividing* both sides of an equation by a factor containing the unknown.

We have already seen the necessity of verifying the results in the solution of equations. In the case of surd equations there is an added reason for verifying, for although there may be no error in the work, the root which is found may not be a root of the given equation.

EXERCISE 145

Solve and verify 1-15. Reject extraneous roots:

1. $\sqrt{2x-5}-3=0.$

2. $\sqrt{3x-2}=2\sqrt{x-2}.$

3. $3x^{\frac{1}{2}}=x^{\frac{1}{2}}+4.$

4. $\sqrt[3]{5x-7}=2.$

5. $2x^{\frac{1}{3}}=3.$

6. $2\sqrt[3]{3x-25}+3=7.$

7. $2(x-7)^{\frac{1}{3}}=(x-14)^{\frac{1}{3}}.$

8. $\sqrt{x}+\sqrt{x+5}=5.$

9. $\sqrt{x+45}+\sqrt{x}=9.$

10. $1+\sqrt{x+2}=\sqrt{x}.$

11. $\sqrt{x+4}+\sqrt{x+15}=11.$

12. $\sqrt{4x^2+3x-16}=2x+2.$

13. $\sqrt{x+5}=\dfrac{x-1}{\sqrt{x-3}}.$

14. $\dfrac{7\sqrt{x}+10}{\sqrt{4x}-2}=4.$

15. $\dfrac{\sqrt{x}-3}{\sqrt{x}+3}=\dfrac{\sqrt{x}+1}{\sqrt{x}-2}.$

16.* $\sqrt{x+4}-\sqrt{x-4}=4.$

17. $(12+x)^{\frac{1}{2}}+x^{\frac{1}{2}}=6.$

18. $(x+3)^{\frac{1}{2}}-(x+3)^{\frac{1}{2}}=2x^{\frac{1}{2}}.$

19. $\dfrac{\sqrt{x}+16}{\sqrt{x}+4}=\dfrac{\sqrt{x}+29}{\sqrt{x}+11}.$

20. $\dfrac{6\sqrt{x}-11}{3\sqrt{x}}=\dfrac{2\sqrt{x}+1}{\sqrt{x}+6}.$

21. $\sqrt{x^2-a^2+b^2}=x-a+b.$

22. $\sqrt{x}+\sqrt{x-9}=\dfrac{36}{\sqrt{x-9}}.$

23. $\sqrt{x-15}+\sqrt{x}=\dfrac{105}{\sqrt{x-15}}.$

24. $\sqrt{x+3}+\sqrt{x+8}=2\sqrt{x}.$

25. $\sqrt{x+4a}-\sqrt{x}=2\sqrt{b+x}.$

26. $\sqrt[3]{x^3-6x^2+11x-5}=x-2.$

27. $5(70x+29)^{\frac{1}{3}}=9(14x-15)^{\frac{1}{3}}.$

28. $\dfrac{\sqrt{x+6}+\sqrt{x-6}}{\sqrt{x+6}-\sqrt{x-6}}=3.$

29. $\dfrac{5x-1}{\sqrt{5x}+1}=1+\dfrac{\sqrt{5x}-1}{2}.$

30. $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=c.$

226. Surd Equations Reducing to Quadratics.

Ex. 1.—Solve $\quad x+\sqrt{x+5}=7.$

Transposing, $\quad \therefore \quad \sqrt{x+5}=7-x.$

Squaring, $\quad \therefore \quad x+5=49-14x+x^2,$

$\therefore \quad x^2-15x+44=0,$

$\therefore \quad (x-4)(x-11)=0,$

$\therefore \quad x=4 \text{ or } 11.$

Verification: When $x=4$, $x+\sqrt{x+5}=4+\sqrt{9}=7,$

When $x=11$, $x+\sqrt{x+5}=11+\sqrt{16}=15,$

$\therefore$ the correct root is 4.

11 is evidently a root of $x-\sqrt{x+5}=7.$

Ex. 2.—Solve $\quad \sqrt{8x+1}-\sqrt{x+1}=\sqrt{3x}.$

Transposing, $\quad \therefore \quad \sqrt{8x+1}=\sqrt{3x}+\sqrt{x+1}.$

Squaring, $\quad \therefore \quad 8x+1=3x+2\sqrt{3x^2+3x}+x+1,$

$\therefore \quad 4x=2\sqrt{3x^2+3x},$

$\therefore \quad 2x=\sqrt{3x^2+3x},$

$\therefore \quad 4x^2=3x^2+3x,$

$\therefore \quad x^2-3x=0,$

$\therefore \quad x=0 \text{ or } 3.$

Here we find on verifying that both roots satisfy the given equation.

Ex. 3.—Solve $\quad 2\sqrt{2x+1}=3-3\sqrt{x-3}.$

Solve as in the preceding and the roots are 4 or 364, neither of which satisfies the equation.

Of what equation is 4 a root? Of what equation is 364 a root?

Ex. 4.—Solve $\quad x^2-3x-6\sqrt{x^2-3x-3}=-2.$

If the surd is removed to one side, we get

$$x^2-3x+2=6\sqrt{x^2-3x-3}.$$

If we now square both sides to remove the surd, we will obtain an equation of the fourth degree which we cannot easily solve.

We may obtain the solution by changing the unknown from x to $\sqrt{x^2-3x-3}$, similar to the method employed in art. 192.

Let
$$\sqrt{x^2-3x-3}=y,$$
$$\therefore \quad x^2-3x-3=y^2,$$
$$\therefore \quad x^2-3x=y^2+3.$$

Substituting in the original equation,
$$y^2+3-6y=-2,$$
$$\therefore \quad y^2-6y+5=0,$$
$$\therefore \quad y=5 \text{ or } 1.$$
$$\therefore \quad x^2-3x-3=25, \qquad \text{or} \qquad x^2-3x-3=1,$$
$$\therefore \quad x^2-3x-28=0, \qquad x^2-3x-4=0,$$
$$\therefore \quad x=7 \text{ or } -4. \qquad x=4 \text{ or } -1.$$

We, therefore, have four roots: 7, -4, 4, -1.

Verify each of these and show that they all satisfy the given equation.

Here both values of y were positive; if either of them had been negative it could at once be discarded as impossible.

Ex. 5.—Solve
$$x+y-\sqrt{x+y}=20, \qquad (1)$$
$$xy-2\sqrt{xy}=120. \qquad (2)$$

From (1), $\quad \sqrt{x+y}=5 \text{ or } -4.$

From (2), $\quad \sqrt{xy}=12 \text{ or } -10.$

Here the negative values of the surds are discarded,
$$\therefore \quad \sqrt{x+y}=5, \quad \sqrt{xy}=12.$$
$$\therefore \quad x+y=25, \quad xy=144.$$

Solving these, $\quad x=9 \text{ or } 16, \; y=16 \text{ or } 9.$

EXERCISE 146

Solve and verify 1-17. Reject extraneous roots:

1. $x+\sqrt{x}=20.$
2. $x-\sqrt{x}=20.$
3. $\sqrt{3x-5}+\sqrt{x-2}=3.$
4. $\sqrt{3x-5}-\sqrt{x-2}=3.$
5. $3x+\sqrt{5x^2+11}+5=0.$
6. $3x+5=\sqrt{5x^2+11}.$
7. $\dfrac{2}{\sqrt{x}}=5-2\sqrt{x}.$
8. $3x-2\sqrt{7x+4}=15.$

9. $\dfrac{\sqrt{x+16}}{\sqrt{4-x}}+\dfrac{\sqrt{4-x}}{\sqrt{x+16}}=\dfrac{5}{2}$. 10. $\sqrt[3]{x^3+2x^2-10x+5}=x-1$.

11. $\sqrt{x+a}+\sqrt{x+b}=\sqrt{a-b}$. 12. $\sqrt{2x+5}-\sqrt{x-1}=2$.

13. $4(x^2+x+3)^{\frac{1}{2}}=3(2x^2+5x-2)^{\frac{1}{2}}$.

14. $3(x+\sqrt{2-x^2})=4(x-\sqrt{2-x^2})$.

15. $\dfrac{\sqrt{3x^2+4}-\sqrt{2x^2+1}}{\sqrt{3x^2+4}+\sqrt{2x^2+1}}=\dfrac{1}{7}$. 16. $\sqrt{\dfrac{x+7}{2}}+\sqrt{\dfrac{x-7}{2}}=7$.

17. $2\sqrt{x}+3\sqrt{y}=12$,
$3\sqrt{x}+2\sqrt{y}=13$.

18.* $xy-\sqrt{xy}=30$,
$x+y=13$.

19. $x+\sqrt{xy}+y=28$,
$x-\sqrt{xy}+y=12$.

20. $x+y+\sqrt{x+y}=30$,
$x-y+\sqrt{x-y}=12$.

21. $x^2+xy+y^2=91$,
$x+\sqrt{xy}+y=13$.

22. $x^2+xy+y^2=5\frac{1}{4}$,
$x-\sqrt{xy}+y=1\frac{1}{2}$

23. $x^2-3x+6-\sqrt{x^2-3x+6}=2$.

24. $x^2-x-\sqrt{x^2-x-6}=36$.

25. $\sqrt{\dfrac{x}{y}}+\sqrt{\dfrac{y}{x}}=\dfrac{5}{2}$, $x+y=10$.

227. Square Root of a Binomial Surd.

$$(\sqrt{3}+\sqrt{2})^2=3+2+2\sqrt{6}=5+2\sqrt{6}.$$
$$(\sqrt{5}-\sqrt{3})^2=5+3-2\sqrt{15}=8-2\sqrt{15}.$$
$$(3-\sqrt{2})^2=9+2-6\sqrt{2}=11-6\sqrt{2}.$$
$$(\sqrt{a}+\sqrt{b})^2=a+b+2\sqrt{ab}.$$
$$(\sqrt{a}-\sqrt{b})^2=a+b-2\sqrt{ab}.$$

The square of $\sqrt{a}+\sqrt{b}$ is made up of a rational quantity $a+b$, which is the sum of the quantities under the root signs, and a surd quantity $2\sqrt{ab}$, the ab being the product of the quantities under the root signs.

The form of the square of $\sqrt{a}+\sqrt{b}$ will show us how we can sometimes find, by inspection, the square root of a binomial surd.

$$\sqrt{a+b+2\sqrt{ab}}=\sqrt{a}+\sqrt{b},$$
$$\sqrt{a+b-2\sqrt{ab}}=\sqrt{a}-\sqrt{b}.$$

Ex. 1.—Find the square root of $7+2\sqrt{12}$.

Here we want two factors of 12, whose sum is 7. They are evidently 4 and 3.

$$\therefore \quad 7+2\sqrt{12}=4+3+2\sqrt{4\,.\,3}.$$
$$\therefore \quad \sqrt{7+2\sqrt{12}}=\sqrt{4}+\sqrt{3}=2+\sqrt{3}.$$

Similarly, $\sqrt{7-2\sqrt{12}}=2-\sqrt{3}$.

Verify by squaring $2+\sqrt{3}$ and $2-\sqrt{3}$.

Ex. 2.—Find the square root of $14-6\sqrt{5}$.

To put this into the form $a+b-2\sqrt{ab}$, first change $6\sqrt{5}$ into $2\sqrt{45}$,

$$\therefore \quad 14-6\sqrt{5}=14-2\sqrt{45}=9+5-2\sqrt{45},$$
$$\therefore \quad \sqrt{14-6\sqrt{5}}=\sqrt{9}-\sqrt{5}=3-\sqrt{5}.$$

EXERCISE 147 (1-7, Oral)

Find the square root and verify:

1. $5+2\sqrt{6}$. 2. $8-2\sqrt{12}$. 3. $4+2\sqrt{3}$.
4. $6-2\sqrt{8}$. 5. $10+2\sqrt{24}$. 6. $15+2\sqrt{56}$.
7. $8+2\sqrt{7}$. 8. $7-4\sqrt{3}$. 9. $9+4\sqrt{5}$.
10. $x+y+2\sqrt{xy}$. 11. $x+y-2\sqrt{xy}$. 12. $2x+2\sqrt{x^2-y^2}$.
13. $15-4\sqrt{14}$. 14. $18-8\sqrt{5}$. 15. $20+\sqrt{300}$.
16. $10+\sqrt{64}$. 17. $47+\sqrt{360}$. 18. $57-18\sqrt{2}$.
19. $a-2\sqrt{a-1}$. 20. $4x+2\sqrt{4x^2-1}$. 21. $a-b-2\sqrt{a-b-1}$.

Ex. 1.—Find the square root of $56-24\sqrt{5}$.

$$56-24\sqrt{5}=56-2\sqrt{720}.$$

Here we require two factors of 720 whose sum is 56. When the numbers are large, as here, it may be difficult to obtain the factors by inspection.

When this is the case we may represent the factors by a and b and find the values of a and b from the equations

$$ab=720,$$
$$a+b=56.$$

Solve these equations by the method of art. 194 or of art. 198 and obtain

$$a=36 \text{ or } 20,$$
$$b=20 \text{ or } 36.$$

The required factors of 720 then are 36 and 20.

$$\therefore\quad 56-24\sqrt{5}=56-2\sqrt{720}=36+20-2\sqrt{36\,.\,20}.$$

$$\therefore\quad \sqrt{56-24\sqrt{5}}=\sqrt{36}-\sqrt{20}=6-2\sqrt{5}.$$

Verify by squaring.

Ex. 2.—Find the square root of $\frac{9}{4}+\sqrt{5}$.

$$\frac{9}{4}+\sqrt{5}=\frac{9+4\sqrt{5}}{4}=\frac{9+2\sqrt{20}}{4}.$$

$\therefore$ the square root is $\dfrac{\sqrt{5}+\sqrt{4}}{2}$ or $\dfrac{\sqrt{5}}{2}+1$.

Ex. 3.—Find the square root of $2\sqrt{10}+6\sqrt{2}$.

First take out the surd factor $\sqrt{2}$, and we get

$$2\sqrt{10}+6\sqrt{2}=\sqrt{2}(6+2\sqrt{5}),$$

$\therefore$ the square root $=\sqrt[4]{2}(1+\sqrt{5})$.

EXERCISE 148

Find the square root and verify:

1. $94-42\sqrt{5}$.
2. $38+12\sqrt{10}$.
3. $47-12\sqrt{15}$.
4. $107-24\sqrt{15}$.
5. $94+6\sqrt{245}$.
6. $101-28\sqrt{13}$.
7. $67+7\sqrt{72}$.
8. $28-5\sqrt{12}$.
9. $xy+2y\sqrt{xy-y^2}$.

10.* Find the value of $1 \div \sqrt{16-6\sqrt{7}}$ to 3 decimal places.

11. Find the value, to three decimals, of the square root of $\dfrac{4-2\sqrt{3}}{7+4\sqrt{3}}$.

12. By first removing a simple surd factor, find the square roots of:

$$7\sqrt{2}+4\sqrt{6},\ 10+6\sqrt{5},\ 7\sqrt{3}-12,\ 59\sqrt{2}+60.$$

13. Show that $\sqrt{17+12\sqrt{2}}+\sqrt{17-12\sqrt{2}}=6$, (1) by taking the square roots, (2) by squaring.

14. Simplify $\sqrt{3+\sqrt{12+\sqrt{49+8\sqrt{3}}}}$.

15. By changing $2-\sqrt{3}$ into $\dfrac{4-2\sqrt{3}}{2}$, find the square root of $2-\sqrt{3}$, also of $\frac{3}{2}+\sqrt{2}$ and of $\frac{11}{3}+\frac{4}{3}\sqrt{7}$.

16. From the result of Ex. 1, show that $94-42\sqrt{5}$ is a positive quantity less than unity.

17. If $x^2(14-6\sqrt{5})=21-8\sqrt{5}$, find x to three decimals.

18. The sides of a right-angled triangle are $\sqrt{5}$ and $3+2\sqrt{2}$. Find the hypotenuse.

228. Imaginary Surds. When we solve the equation $x^2=9$, we obtain $x=\pm 3$, and we know that this is the correct result, for the square of either $+3$ or -3 is 9.

If we solve $x^2=5$, we say that the value of x is $\pm\sqrt{5}$, and we can approximate to the values of the roots as closely as we wish by finding the square root of 5 by the formal method.

If we are asked to solve $x^2=-9$, we might say that the solution is impossible, as there is no number whose square is -9. This statement is correct, but we find it convenient to say

if $\qquad x^2=-9,$

then $\qquad x=\pm\sqrt{-9}.$

Such a quantity as $\sqrt{-9}$ is called an **imaginary quantity**, and must be distinguished from such quantities as 5, $-\frac{1}{4}$, $\frac{1}{3}\sqrt{7}$, etc., which are called **real quantities**.

We may define an imaginary quantity as one whose square is negative, or as the square root of a negative quantity.

We have already seen how imaginary quantities sometimes appear in the solution of quadratic equations (art. 190).

We will assume that the fundamental laws of algebra, which we have applied in using real numbers, apply also to imaginary numbers.

Thus,

$$\sqrt{-9}=\sqrt{9}\times\sqrt{-1}=3\sqrt{-1}.$$

$$\sqrt{-25}+\sqrt{-4}=5\sqrt{-1}+2\sqrt{-1}=7\sqrt{-1}.$$

$$\sqrt{-a^2}=\sqrt{a^2}\times\sqrt{-1}=a\sqrt{-1}.$$

$$\sqrt{-6}=\sqrt{6}\times\sqrt{-1}.$$

These examples show that an imaginary quantity can always be expressed as the product of a real quantity and the imaginary quantity $\sqrt{-1}$. The quantity $\sqrt{-1}$ is sometimes called the **imaginary unit**.

229. Powers of the Imaginary Unit. Any even power of $\sqrt{-1}$ is real, and any odd power is imaginary.

Thus,

$$(\sqrt{-1})^2=-1, \text{ by definition,}$$

$$\therefore\ (\sqrt{-1})^3=-\sqrt{-1},$$

$$\therefore\ (\sqrt{-1})^4=(-1)^2=+1,$$

$$\therefore\ (\sqrt{-1})^5=(\sqrt{-1})^4\times\sqrt{-1}=+\sqrt{-1}, \text{ etc.}$$

230. Multiplication and Division of Imaginaries.

Ex. 1. $\sqrt{-2}\times\sqrt{-3}=\sqrt{2}\,.\,\sqrt{-1}\times\sqrt{3}\,.\,\sqrt{-1},$

$$=\sqrt{2}\,.\,\sqrt{3}\times(\sqrt{-1})^2=-\sqrt{6}.$$

Note that the product here is $-\sqrt{6}$, not $\sqrt{6}$.

Ex. 2. $3\sqrt{-4}\times5\sqrt{-9}=6\sqrt{-1}\times15\sqrt{-1},$

$$=90(\sqrt{-1})^2=-90.$$

Ex. 3. $\dfrac{\sqrt{-18}}{\sqrt{-2}}=\dfrac{\sqrt{18}\times\sqrt{-1}}{\sqrt{2}\times\sqrt{-1}}=\dfrac{\sqrt{18}}{\sqrt{2}}=\sqrt{9}=3.$

Ex. 4. $(x+y\sqrt{-1})^2=x^2+y^2(\sqrt{-1})^2+2xy\sqrt{-1},$

$$=x^2-y^2+2xy\sqrt{-1}.$$

Ex. 5. $(a+b\sqrt{-1})(a-b\sqrt{-1})=a^2-b^2(\sqrt{-1})^2=a^2+b^2.$

Ex. 6. Rationalize the denominator of $\dfrac{3}{1-\sqrt{-2}}$.

Multiply both terms by $1+\sqrt{-2}$ and we get

$$\frac{3(1+\sqrt{-2})}{(1-\sqrt{-2})(1+\sqrt{-2})}=\frac{3(1+\sqrt{-2})}{1-(-2)}=1+\sqrt{-2}.$$

EXERCISE 149 (1-9, Oral)

1. Express as a multiple of $\sqrt{-1}$: $\sqrt{-4}$, $\sqrt{-16}$, $\sqrt{-81}$, $\sqrt{-a^2}$ $\sqrt{-625}$, $\sqrt{-9x^4}$, $\sqrt{-(a-b)^2}$.

2. What is the value of $(\sqrt{-1})^2$, $(\sqrt{-a})^2$, $(\sqrt{-1})^4$, $(\sqrt{-1})^6$?

Find the sum of :

3. $\sqrt{-1}$, $\sqrt{-4}$, $\sqrt{-9}$.
4. $\sqrt{-25}$, $\sqrt{-100}$, $\sqrt{-49}$.
5. $4+\sqrt{-9}$, $2-\sqrt{-16}$.
6. $a+\sqrt{-b^2}$, $a-b\sqrt{-1}$.

What is the product of :

7. $\sqrt{-1}$, $\sqrt{-4}$.
8. $\sqrt{-25}$, $\sqrt{-100}$.
9. $\sqrt{-a^2}$, $\sqrt{-b^2}$

Simplify :

10.* $3\sqrt{-3}+2\sqrt{-75}-4\sqrt{-12}+5\sqrt{-48}$.

11. $(3+5\sqrt{-1})(3-5\sqrt{-1})+(5-3\sqrt{-1})(5+3\sqrt{-1})$.

12. $(4-3\sqrt{-1})^2+(2+6\sqrt{-1})^2$.

13. $2\div(1-\sqrt{-1})$.

14. $(-1+\sqrt{-3})\div(-1-\sqrt{-3})$.

15. $(a+b\sqrt{-1})^2+(a-b\sqrt{-1})^2$.

16. Show that $\frac{1}{4}(-1+\sqrt{-3})^2=\frac{1}{2}(-1-\sqrt{-3})$.

17. By finding the cube of $\frac{1}{2}(-1+\sqrt{-3})$, show that this quantity is a cube root of unity. (art. 192, Ex. 5.)

18. Are $2 \pm \sqrt{-3}$, the roots of $x^2-4x+7=0$?

19. If $a=2+3\sqrt{-1}$ and $b=2-3\sqrt{-1}$, show that $a+b$, ab and a^2+b^2 are real quantities.

231. Impossible Problems. We obtained the imaginary number $\sqrt{-9}$ in answer to the question, "What is the number whose square is -9?" As we have said, this is arithmetically an impossible problem.

When we obtain an imaginary result in solving a problem, we may conclude that the problem is impossible.

Ex. 1.—The sum of a number and its reciprocal is $1\frac{1}{2}$. Find the number.

Let x = the number, $\therefore \quad \frac{1}{x}$ = its reciprocal,

$$\therefore \quad x+\frac{1}{x}=1\frac{1}{2},$$

$$\therefore \quad 2x^2-3x+2=0,$$

$$\therefore \quad x=\frac{3\pm\sqrt{9-16}}{4}=\frac{3\pm\sqrt{-7}}{4}.$$

Here the roots are imaginary, and we conclude that there is no number which answers the condition of the problem.

In fact, it may be shown that the sum of a positive number and its reciprocal is never less than 2.

Change $1\frac{1}{2}$ into $2\frac{1}{2}$ and solve the problem.

Ex. 2.—For \$30 I can buy x yards of cloth at \$$(10-x)$ per yard. Find x.

The total cost in dollars $=x(10-x)=30$.

$$\therefore \quad x^2-10x+30=0,$$

$$\therefore \quad x=\frac{10\pm\sqrt{-20}}{2}.$$

What conclusion do you draw? Would it be impossible if for \$30 we substitute \$25? \$20?

EXERCISE 150

Solve and determine if these problems are possible:

1. A line which is 10 inches long is divided into two parts so that the area of the rectangle contained by the parts is 40 square inches. Find the lengths of the parts.

2. The length of a rectangle is twice its width. If the length be increased 10 feet and the width decreased 1 foot, the area is doubled. Find the dimensions. Solve also when the width is increased 1 foot.

3. A man has 20 miles to walk. If he walks at x miles per hour it will take him $8-x$ hours. At what rate does he walk?

4. If it is possible that $x(12-x)=36+a$, and a is not negative, what must the value of a be?

EXERCISE 151 (Review of Chapter XXIV)

1.* Multiply $1+\sqrt{3}-\sqrt{2}$ by $\sqrt{6}-\sqrt{2}$.

2. Multiply $\sqrt{3}+\sqrt{2}$ by $\dfrac{1}{\sqrt{3}}+\dfrac{1}{\sqrt{2}}$.

3. Find the square roots of: $14+\sqrt{180}$, $25-4\sqrt{21}$, $22+\sqrt{420}$, $11-\sqrt{72}$, $12-6\sqrt{3}$.

4. Find to two decimal places the values of:

$$\frac{1}{\sqrt{5}-\sqrt{3}},\quad \sqrt{5+2\sqrt{6}},\quad \frac{7+\sqrt{3}}{2\sqrt{3}+\sqrt{8}},\quad \frac{3\sqrt{8}+\sqrt{20}}{\sqrt{5}+\sqrt{2}}.$$

Solve and verify:

5. $x+\sqrt{x-5}=11$.

6. $\sqrt{4x+7}+\sqrt{4x+3}=6$.

7. $\sqrt{x}+\sqrt{x-4}=\dfrac{8}{\sqrt{x-4}}$.

8. $\sqrt{6x+7}-\sqrt{x+2}=\sqrt{2}\sqrt{x+1}$.

9. $\sqrt{x+5}+\sqrt{x-16}=\dfrac{35}{\sqrt{x+5}}$.

10. $\sqrt{19+x}-\sqrt{15-x}=\dfrac{24}{\sqrt{19+x}}$.

11. Multiply $\sqrt{2}+\sqrt{3}+\sqrt{5}$ by $\sqrt{5}+\sqrt{3}-\sqrt{2}$ and

$$x+y+2\sqrt{x+y} \text{ by } x+y-2\sqrt{x+y}.$$

12. When $x=2+\sqrt{3}$, $y=2-\sqrt{3}$, find the value of

$$\frac{2x+y}{x-y}+\frac{x-2y}{x+y}.$$

13. Expand and simplify $(\sqrt{2a+\sqrt{b}}+\sqrt{2a-\sqrt{b}})^2$. Check the result by substituting $a=13$, $b=100$.

14. Solve $p+x-\sqrt{2px+x^2}=q$.

15. Find the product of $2\sqrt{3}+3\sqrt{2}+\sqrt{30}$ and $\sqrt{2}+\sqrt{3}-\sqrt{5}$.

16. Find the continued product of

$$x-1+\sqrt{2},\ x-1-\sqrt{2},\ x+1+\sqrt{3} \text{ and } x+1-\sqrt{3}.$$

17. Simplify $\dfrac{3\sqrt{2}}{\sqrt{6}+\sqrt{3}}-\dfrac{2\sqrt{6}}{\sqrt{3}+1}+\dfrac{\sqrt{6}}{\sqrt{3}+\sqrt{2}}$.

18. When $x=a+\sqrt{a^2-1}$, find the values of

$$x+\frac{1}{x},\quad x^2+\frac{1}{x^2},\quad x^3+\frac{1}{x^3}.$$

19. Express in the simplest form:

$$\sqrt{27}-\sqrt{8}+\sqrt{17+12\sqrt{2}}-\sqrt{28-6\sqrt{3}}$$

and

$$\sqrt{11+6\sqrt{2}}+\sqrt{19-4\sqrt{12}}+\sqrt{5-\sqrt{24}}.$$

20. Simplify $\dfrac{\sqrt{12+6\sqrt{3}}}{\sqrt{3}+1}$ and $\sqrt{\dfrac{m+n}{m-n}}+\sqrt{\dfrac{m-n}{m+n}}$.

21. If $x=-1+2\sqrt{-1}$, find the value of x^4-12x.

22. Find the square roots of $7+\sqrt{13}$ and $2a+\sqrt{4a^2-4}$.

23. Solve $2x^2+6x-6-\sqrt{x^2+3x-3}=45$.

24. Simplify $(\sqrt{5}+\sqrt{3}+\sqrt{2})^2+(\sqrt{5}+\sqrt{3}-\sqrt{2})^2$

$$+(\sqrt{5}-\sqrt{3}+\sqrt{2})^2+(\sqrt{5}-\sqrt{3}-\sqrt{2})^2.$$

25. Solve $3x^2-9x+11=4\sqrt{x^2-3x+5}$, giving the roots to two decimal places.

26. Simplify $\dfrac{\sqrt{a+b}+\sqrt{a-b}}{\sqrt{a+b}-\sqrt{a-b}}-\dfrac{\sqrt{a+b}-\sqrt{a-b}}{\sqrt{a+b}+\sqrt{a-b}}$.

27. Show that $\dfrac{1}{\sqrt{16+2\sqrt{63}}}+\dfrac{1}{\sqrt{16-2\sqrt{63}}}=3$.

28. Show that $\sqrt{a+\sqrt{b}}$ cannot be expressed in the form $\sqrt{x}+\sqrt{y}$ unless a^2-b is a perfect square.

29. Simplify $\left(\dfrac{a^x+a^{-x}}{2}\right)^2+\left(\dfrac{a^x-a^{-x}}{2\sqrt{-1}}\right)^2$.

30. Simplify $(3-2\sqrt{2})^{\frac{3}{2}}+(3+2\sqrt{2})^{\frac{3}{2}}$.

31. Find the value of x^3+x^2+x+1 when $x=\sqrt{2}+1$.

CHAPTER XXV

THEORY OF QUADRATICS

232. Sum and Product of the Roots.

Solve these equations:

(1) $x^2-11x+10=0.$ The roots are 10, 1.
(2) $2x^2-\ 3x-\ 5=0.$,, ,, ,, $\frac{5}{2}$, -1.
(3) $15x^2+26x+\ 8=0.$,, ,, ,, $-\frac{2}{5}$, $\frac{4}{3}$.

In (1) the sum of the roots = 11, product = 10.
In (2) ,, ,, ,, ,, ,, = $\frac{3}{2}$, ,, $=-\frac{5}{2}$.
In (3) ,, ,, ,, ,, ,, $=-\frac{26}{15}$, ,, $=\frac{8}{15}$.

Examine the sum and the product in each case and state how they compare with the coefficients in the given equations.

Every quadratic equation may be reduced to the form

$$ax^2+bx+c=0.$$

The roots are $\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $\dfrac{-b-\sqrt{b^2-4ac}}{2a}$.

For brevity represent these roots by m and n,

$$\therefore \quad m+n=\frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}=\frac{-2b}{2a}=-\frac{b}{a},$$

and
$$mn=\frac{(-b+\sqrt{b^2-4ac})(-b-\sqrt{b^2-4ac})}{4a^2},$$

$$=\frac{(-b)^2-(\sqrt{b^2-4ac})^2}{4a^2}=\frac{b^2-b^2+4ac}{4a^2}=\frac{4ac}{4a^2}=\frac{c}{a}.$$

Comparing these results with the coefficients a, b, c in the equation, we see that :

The sum of the roots of any quadratic equation, in the standard form, is equal to the coefficient of x *with its sign changed, divided by the coefficient of* x^2, *and the product of the roots is equal to the absolute term, divided by the coefficient of* x^2.

$$\text{Sum of the roots} = -\frac{\text{coefficient of } x}{\text{coefficient of } x^2}.$$

$$\text{Product of the roots} = \frac{\text{absolute term}}{\text{coefficient of } x^2}.$$

See if these two statements apply to the roots and coefficients of the three equations preceding.

The formulæ for the sum and product of the roots furnish a convenient means of verifying the roots.

Thus, I find the roots of $3x^2+x-2=0$ to be $\frac{2}{3}$, -1, but the sum of $\frac{2}{3}$ and -1 is $-\frac{1}{3}$ and the product is $-\frac{2}{3}$, which agree with the sum and product given by the formulæ. Therefore, these are the correct roots.

Are the roots of $14x^2-19x-60=0$, $\frac{20}{7}$, $-\frac{3}{2}$?

233. Reciprocal Roots. If the roots of $ax^2+bx+c=0$ are reciprocals (like $\frac{2}{3}$ and $\frac{3}{2}$), their product is unity, and therefore $\frac{c}{a}=1$ or $c=a$.

So that *any quadratic equation, in which the coefficient of* x^2 *is equal to the absolute term, will have reciprocal roots.*

Thus, the roots of $6x^2-13x+6=0$ are reciprocals, since their product is $\frac{6}{6}$ or 1. Verify this by finding the roots.

234. Roots equal in Magnitude but opposite in Sign. If the roots of $ax^2+bx+c=0$ are equal in magnitude but opposite in sign (like 3 and -3), their sum will be zero, therefore $-\frac{b}{a}=0$ or $b=0$.

So that *any quadratic equation in which the second term is missing will have roots equal in magnitude but opposite in sign.*

Thus, $2x^2-9=0$ and $ax^2-c=0$ have such roots.

Verify by finding the roots.

EXERCISE 152 (Oral)

State the sum and product of the roots of :

1. $x^2-7x+12=0.$

2. $x^2-5x-11=0.$

3. $x^2+6x+1=0.$

4. $2x^2-10x+6=0.$

5. $3x^2-12x-7=0.$

6. $4x^2-17x+4=0.$

7. $ax^2-bx-c=0.$

8. $ax^2-(b+c)x+a=0.$

9. $px^2-q=0.$

10. $ax^2+a=0.$

11. $3x^2-4x=6.$

12. $(a+b)x^2-x+a^2-b^2=0.$

13. Which of the preceding equations have reciprocal roots ? Which have roots equal in magnitude but opposite in sign ?

14. Are 4 and 5 the roots of $x^2-9x+20=0$?

15. Are $3+\sqrt{2}$, $3-\sqrt{2}$ the roots of $x^2-6x+7=0$?

In which of the following are the correct roots given :

16. $x^2-7x+10=0$; 5, 2.

17. $x^2+3x-28=0$; 7, -4.

18. $x^2-13x+36=0$; 4, 9.

19. $x^2-12x+27=0$; 4, 8.

20. $x^2-4x-5=0$; 5, 1.

21. $2x^2-5x+2=0$; 2, $\frac{1}{2}$.

22. In solving $x^2-2x-1599=0$, one root is found to be 41. What must the other be ?

23. How would you show that 1·125 and 2·168 are the correct roots of $x^2-3{\cdot}293x+2{\cdot}439=0$?

24. If the roots of $6x^2-10x+a=0$ are reciprocals, what is the value of a ?

25. If the roots of $mx^2-(m^2-9)x+m^2=0$ are equal in magnitude but opposite in sign, what is the value of m ? What would then be the product of the roots ?

235. To form a Quadratic with given Roots. First Method. In the equation $x^2+px+q=0$, the sum of the roots is $-p$, and the product is q. Since every quadratic equation may be reduced to the form $x^2+px+q=0$, by dividing by the coefficient of x^2, any quadratic equation may be written thus :

$$x^2-x\,(\textbf{sum of roots})+(\textbf{product of roots})=0.$$

If the roots are given, the equation can at once be written down.

Thus, the equation whose roots are 3 and 5 is $x^2-x(3+5)+3.5=0$, or $x^2-8x+15=0$

The equation whose roots are $2+\sqrt{3}$ and $2-\sqrt{3}$ is

$$x^2-x(2+\sqrt{3}+2-\sqrt{3})+(2+\sqrt{3})(2-\sqrt{3})=0, \text{ or } x^2-4x+1=0.$$

The equation whose roots are $a+b$ and $a-b$ is $x^2-2ax+a^2-b^2=0$.

Second Method. The equation whose roots are p and q is $(x-p)(x-q)=0$.

The equation whose roots are 3 and 5 is $(x-3)(x-5)=0$, or $x^2-8x+15=0$.

The equation whose roots are $\frac{2}{3}$ and $-\frac{3}{4}$ is $(x-\frac{2}{3})(x+\frac{3}{4})=0$, or $(3x-2)(4x+3)=0$.

The equation whose roots are $2+\sqrt{3}$, $2-\sqrt{3}$ is

$$(x-2-\sqrt{3})(x-2+\sqrt{3})=0, \text{ or } (x-2)^2-3=0 \text{ or } x^2-4x+1=0.$$

Either method is simple enough to apply, but the first is probably easier when the given roots are not simple numbers.

The second method may be applied to form an equation with any number of given roots.

Thus, the equation whose roots are 2, 3, -5 is

$$(x-2)(x-3)(x+5)=0, \text{ or } x^3-19x+30=0.$$

EXERCISE 158 (1-16, Oral)

State, without simplifying, the equations whose roots are:

1. 3, 7.
2. 3, -7.
3. -3, 7.
4. -3, -7
5. $\frac{1}{2}$, $\frac{1}{3}$.
6. $\frac{1}{2}$, -1.
7. $-\frac{1}{4}$, $-\frac{1}{3}$.
8. $\frac{2}{3}$, $\frac{3}{4}$.
9. a, a.
10. $-a$, $-b$.
11. 3, 0.
12. 0, m.
13. 3, 4, 5.
14. 2, 3, -1.
15. a, b, c.
16. a, b, 0.

Reduce to the simplest form the equations whose roots are:

17.* $m+n$, $m-n$.

18. $2a-b$, $2a+b$.

19. $3+\sqrt{3}$, $3-\sqrt{3}$

20. $1\frac{2}{3}$, $-2\frac{1}{4}$.

21. -2, -4, 6.

22. $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$.

23. Show that 1·25 and 4·64 are the correct roots of

$$100x^2-589x+580=0.$$

24. Construct an equation in which the sum of the roots is 7 and the difference of their squares is 14.

25. Form the equation whose roots are a and b where $a^2+b^2=25$, $a+b=7$.

26. Form the equations whose roots are

$$\frac{a+b}{a-b},\ \frac{a-b}{a+b};\quad \tfrac{1}{2}(4\pm\sqrt{7}).$$

27. Find the sum and the product of the roots of:

(1) $(x-2)^2=5x-3$.
(2) $(x-a)(x-b)=ab$.
(3) $x(x-p)=p(x-q)$.
(4) $(x+a)^2+(x+b)^2=(x+c)^2$.

28. Solve $x^4-21x^2-20x=0$, being given that one root is 5.

29. If one root of $x^2-12x+a=0$ is double the other, find the roots and the value of a.

30. If one root of $x^2+px+48=0$ is three times the other, what are the values of p?

236. Functions of the Roots. When m and n are the roots of $ax^2+bx+c=0$,

$$m+n=-\frac{b}{a},\ mn=\frac{c}{a}.$$

Here it will be seen, that the sum and the product of the roots do not contain surd expressions, while the separate roots do.

If we wish to find the sum of the squares of the roots, we can do so in the following way:

$$\begin{aligned} m^2+n^2 &= (m+n)^2-2mn,\\ &=\left(-\frac{b}{a}\right)^2-2.\frac{c}{a},\\ &=\frac{b^2}{a^2}-\frac{2c}{a}=\frac{b^2-2ac}{a^2}. \end{aligned}$$

It can also be found by taking the square of each root and adding the results. Find it that way and see if you get the same result.

Ex. 1.—When m and n are the roots of $ax^2+bx+c=0$, find the values of $\frac{1}{m}+\frac{1}{n}$, $\frac{m}{n}+\frac{n}{m}$, m^3+n^3, $m-n$.

$$\frac{1}{m}+\frac{1}{n}=\frac{m+n}{mn}=-\frac{b}{a}\div\frac{c}{a}=-\frac{b}{c}.$$

$$\frac{m}{n}+\frac{n}{m}=\frac{m^2+n^2}{mn}=\frac{(m+n)^2-2mn}{mn}=\frac{\frac{b^2}{a^2}-\frac{2c}{a}}{\frac{c}{a}}=\frac{b^2-2ac}{ac}.$$

$$m^3+n^3=(m+n)^3-3mn(m+n)=-\frac{b^3}{a^3}+\frac{3bc}{a^2}=\frac{3abc-b^3}{a^3},$$

or $$m^3+n^3=(m+n)(m^2-mn+n^2)=(m+n)\{(m+n)^2-3mn\}=\text{etc.}$$

$$(m-n)^2=(m+n)^2-4mn=\frac{b^2}{a^2}-\frac{4c}{a}=\frac{b^2-4ac}{a^2},$$

$$\therefore \quad m-n=\pm\frac{\sqrt{b^2-4ac}}{a}.$$

The same two values of the last expression might have been found by simple subtraction, the sign depending on the order in which the roots were taken.

Ex. 2.—If m and n are the roots of $x^2+px+q=0$, find the equation whose roots are m^2 and n^2.

Here $m+n=-p$ and $mn=q$.

The sum of the roots of the required equation is

$$m^2+n^2=(m+n)^2-2mn=p^2-2q.$$

The product of the roots $=m^2n^2=q^2$.

$\therefore$ the required equation is $x^2-x(p^2-2q)+q^2=0$.

Ex. 3.—Find the equation whose roots are each greater by 2 than the roots of $6x^2-13x-8=0$.

Solve the given equation and the roots are $\frac{8}{3}$, $-\frac{1}{2}$.

$\therefore$ the roots of the required equation are $\frac{14}{3}$, $\frac{3}{2}$.

$\therefore$ the required equation is

$$x^2-(\tfrac{14}{3}+\tfrac{3}{2})x+\tfrac{14}{3}\cdot\tfrac{3}{2}=0, \text{ or } 6x^2-37x+42=0.$$

We might have solved the problem without finding the actual roots of the given equation.

Let p and q be the roots of $6x^2-13x-8=0$.

Then
$$p+q=\tfrac{13}{6} \text{ and } pq=-\tfrac{4}{3}.$$

$\therefore$ the sum of the roots of the required equation is
$$p+2+q+2=p+q+4=\tfrac{13}{6}+4=\tfrac{37}{6},$$
and the product
$$=(p+2)(q+2)=pq+2(p+q)+4=-\tfrac{4}{3}+\tfrac{13}{3}+4=7.$$

$\therefore$ the required equation is
$$x^2-\tfrac{37}{6}x+7=0, \text{ or } 6x^2-37x+42=0.$$

When would the second method be simpler than the first?

Ex. 4.—Find the equation whose roots are the reciprocals of the roots of $mx^2+nx+k=0$.

Let p and q be the roots of the given equation,

then
$$p+q=-\frac{n}{m} \text{ and } pq=\frac{k}{m}.$$

The roots of the required equation are $\dfrac{1}{p}$ and $\dfrac{1}{q}$.

Find the sum and product of $\dfrac{1}{p}$ and $\dfrac{1}{q}$ in terms of m, n and k and complete the solution.

Compare the new equation with the given one and see if you could not write down, mentally, the equation whose roots are the reciprocals of the roots of any given equation.

237. The following method will be found useful in solving such problems as the three preceding.

Ex. 1.—Find the equation whose roots are each greater by 5 than the roots of $4x^2-5x+7=0$.

Let y be the unknown in the required equation.

Then
$$y=x+5 \text{ or } x=y-5.$$

Substitute $x=y-5$ in the given equation, and the required equation is
$$4(y-5)^2-5(y-5)+7=0,$$
or
$$4y^2-40y+100-5y+25+7=0,$$
or
$$4y^2-45y+132=0.$$

Ex. 3.—Find the equation whose roots are the squares of the roots of $ax^2+bx+c=0$.

Let y be the unknown in the required equation.

Then $$y=x^2 \text{ or } x=\pm\sqrt{y},$$

$\therefore$ the required equation is $a(\pm\sqrt{y})^2+b(\pm\sqrt{y})+c=0$,

or $$ay+c=\mp b\sqrt{y},$$

or $$a^2y^2+2acy+c^2=b^2y,$$

or $$a^2y^2+y(2ac-b^2)+c^2=0.$$

Solve Ex.'s 2, 3, 4 preceding, by this method.

EXERCISE 154

1.* If m and n are the roots of $x^2-5x+3=0$, find the values of $\frac{1}{m}+\frac{1}{n}$, $\frac{m}{n}+\frac{n}{m}$, m^2+mn+n^2.

2. Find the sum of the squares of the roots of

$$x^2-7x+1=0 \text{ and of } 3x^2-4x+5=0.$$

3. If p and q are the roots of $3x^2+2x-6=0$, find the values of $\frac{p}{q}+\frac{q}{p}$, $\frac{1}{p^2}+\frac{1}{q^2}$, p^2-pq+q^2.

4. Find the sum of the cubes of the roots of

$$2x^2-3x+4=0 \text{ and of } x^2-x+a=0.$$

5. Find the equation whose roots are double the roots of $x^2-9x+20=0$, (1) by solving, (2) without solving.

6. Find the equations whose roots are each less by 3 than the roots of (1) $x^2-11x+28=0$, (2) $x^2-x-1=0$.

7. Find the equations whose roots are the reciprocals of the roots of (1) $2x^2+x-6=0$, (2) $x^2-px+q=0$.

8. If m and n are the roots of $3x^2-2x+5=0$, find the equations whose roots are :

$$(1)\ \frac{1}{m} \text{ and } \frac{1}{n},\quad (2)\ \frac{m}{n} \text{ and } \frac{n}{m},\quad (3)\ m^2 \text{ and } n^2.$$

9. Find the sum of the squares and the sum of the cubes of the roots of $x^2+ax+b=0$.

10. Find the equation whose roots are the squares of the roots of $x^2+px-q=0$.

11. Find the equation whose roots are each greater by h than the roots of $ax^2+bx+c=0$.

12 Find the equation whose roots are the reciprocals of the roots of $x^2+x=\frac{1}{2}$.

13. If m and n are the roots of $x^2-px+q=0$, show that $m+n$ and mn are the roots of $x^2-x(p+q)+pq=0$.

14. Form the equation whose roots are m and n, where

$$m^2+n^2=20,\ m+n=-6.$$

15. If m and n are the roots of $x^2+px+q=0$, show that $m+2n$ and $2m+n$ are the roots of $x^2+3px+2p^2+q=0$.

16. If p and q are the roots of $ax^2+bx+c=0$, find the value of $p^4+p^2q^2+q^4$ in terms of a, b and c.

238. Character of the Roots of a Quadratic Equation.

Solve the equations :

(1) $x^2-6x+9=0$, the roots are 3, 3.

(2) $6x^2+x-15=0$, ,, ,, ,, $\frac{3}{2}, -\frac{5}{3}$.

(3) $5x^2+7x-2=0$, ,, ,, ,, $\dfrac{-7\pm\sqrt{89}}{10}$.

(4) $2x^2-3x+2=0$, ,, ,, ,, $\dfrac{3\pm\sqrt{-7}}{4}$.

In (1), the roots are equal. We might say that there is only one root, but we prefer to say that there are two roots, which in this case happen to be equal.

In (3), the roots are irrational, but we can approximate to their values by taking the square root of 89.

In (4), the roots are also irrational, but we can not even approximate to their values. Here the roots are imaginary, while in each of the others the roots are real.

These statements might be written thus:

In (1), *the roots are equal, real and rational.*

In (2), *the roots are unequal, real and rational.*

In (3), *the roots are irrational and real.*

In (4), *the roots are irrational and imaginary.*

If we examine the roots of the general quadratic equation we will see the reason why, under particular conditions, there is this difference in the character of the roots.

The roots of $ax^2+bx+c=0$ are

$$\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \frac{-b-\sqrt{b^2-4ac}}{2a}.$$

From these roots we may conclude:

(1) If the particular values of a, b, c are such that $b^2-4ac=0$, then the roots are equal, for each is evidently equal to $-\frac{b}{2a}$

In equation (1), $a=1$, $b=-6$, $c=9$.

$$\therefore \quad b^2-4ac=36-36=0.$$

(2) If b^2-4ac is a perfect square, then its square root can be found exactly and the roots are rational.

In equation (2), $a=6$, $b=1$, $c=-15$.

$$\therefore \quad b^2-4ac=1+360=361=19^2.$$

(3) If b^2-4ac is not a perfect square, but is positive, the roots are real but irrational.

Find the value of b^2-4ac in equation (3).

(4) If b^2-4ac is negative, the roots are imaginary.

Find the value of b^2-4ac in equation (4).

Hence, *the roots of* $ax^2+bx+c=0$ *are real and equal if* $b^2-4ac=0$, *real and unequal if* b^2-4ac *is positive, imaginary if* b^2-4ac *is negative, real and rational if* b^2-4ac *is a perfect square.*

239. The Discriminant. We see then, that we can determine the character of the roots of a quadratic equation without actually finding the roots. All we require to do is to find the value of b^2-4ac.

This important quantity is called the **discriminant** of the equation $ax^2+bx+c=0$.

Ex. 1.—Determine the character of the roots of:

(1) $3x^2+5x-11=0$. (2) $12x^2-25x+12=0$.
(3) $x^2-x+3=0$. (4) $2x^2-16x+32=0$.

The value of the discriminant (b^2-4ac)

in (1) is 157, ∴ the roots are real and irrational,
in (2) is 49, ∴ the roots are real and rational,
in (3) is -11, ∴ the roots are imaginary,
in (4) is 0, ∴ the roots are real and equal.

Ex. 2.—For what values of k will $4x^2-kx+4=0$ have equal roots?

The roots will be equal if $b^2-4ac=0$,
that is, if $k^2-64=0$ or if $k=\pm 8$.

Substitute these values for k and see if the roots are equal.

Ex. 3.—Show that the roots are rational of

$$3mx^2-x(2m+3n)+2n=0.$$

Here
$$b^2-4ac=(2m+3n)^2-24mn,$$
$$=4m^2-12mn+9n^2=(2m-3n)^2.$$

Since b^2-4ac is a square, the roots are rational. Verify by finding the roots.

EXERCISE 155 (1-5, Oral)

1. What is the discriminant of $x^2+4x+4=0$? What is the character of the roots?

2. What is the nature of the roots of $x^2+3x+2=0$?

3. What is peculiar about the roots if $b^2-4ac=0$?

4. What kind of roots have $x^2-5x+7=0$, $x^2-6x+9=0$,
$x^2-x-6=0$, $x^2-4x-6=0$?

5. If the discriminant is ·25, what is the character of the roots?

Determine the character of the roots of:

6.* $2x^2+5x+3=0$.

7. $3x^2-7x-5=0$.

8. $4x^2+7x+15=0$.

9. $9x^2-12x+4=0$.

10. $abx^2+x(a^2+b^2)+ab=0$.

11. $x^2-mx-1=0$.

12. Show that $x^2+ax+b=0$ has real roots for all negative values of b.

13. If $9x^2+12x+k=0$ has equal roots, find k.

14. If $ax^2-10x+a=0$ has equal roots, find a.

15. Show that the roots of $x^2-x(1+k)+k=0$ are rational for all values of k.

16. If $x^2+2x(1+a)+a^2=0$ has equal roots, find a.

17. By solving the equation $x^2-4x+5=k$, show that if x is real, k cannot be less than 1.

18. Show that the roots of $\dfrac{1}{x}+\dfrac{1}{x+a}+\dfrac{1}{x+b}=0$ are real if a^2-ab+b^2 is positive.

19. Eliminate y from the equations $y=mx+c$ and $y^2=4ax$, and find the value of c if the resulting equation in x has equal roots.

20. If $2mx^2+(5m+2)x+(4m+1)=0$ has equal roots, find the values of m and verify.

240. Factors of a Quadratic Expression.

When m and n are the roots of $ax^2+bx+c=0$,

$$m+n=-\frac{b}{a},\quad mn=\frac{c}{a}.$$

$$\begin{aligned}\therefore\quad ax^2+bx+c&=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)\\&=a\{x^2-(m+n)x+mn\}\\&=a(x-m)(x-n).\end{aligned}$$

So that, if m and n are the roots of $ax^2+bx+c=0$, the factors of the quadratic expression

$$ax^2+bx+c \text{ are } a(x-m)(x-n).$$

We can, therefore, find the factors of a trinomial like ax^2+bx+c by solving the corresponding equation.

Ex. 1.—Factor $6x^2+x-40$.

Solving by formula, we find the roots of

$$6x^2+x-40=0 \text{ are } \tfrac{5}{2},\ -\tfrac{8}{3}.$$

$$\therefore\quad 6x^2+x-40=6(x-\tfrac{5}{2})(x+\tfrac{8}{3})$$
$$=(2x-5)(3x+8).$$

Ex. 2.—Factor $12x^2-47x+40$.

The roots of the corresponding equation are $\frac{5}{4}$, $\frac{8}{3}$.

$$\therefore\quad 12x^2-47x+40=12(x-\tfrac{5}{4})(x-\tfrac{8}{3})$$
$$=(4x-5)(3x-8).$$

241. Character of the Factors of a Trinomial. Since $ax^2+bx+c=0$ has equal roots when $b^2-4ac=0$, it follows that $\mathrm{ax^2+bx+c}$ *has equal factors, or is a square, when* $\mathrm{b^2-4ac=0}$.

Thus, in $3x^2-30x+75$, $b^2-4ac=900-900=0$.

$\therefore$ $3x^2-30x+75$ is a perfect square when the numerical factor 3 is removed.

If b^2-4ac is a perfect square, the expression ax^2+bx+c has two rational factors, for under this condition the corresponding equation has rational roots.

Thus, in $20x^2-x-12$, $b^2-4ac=961=31^2$.

$\therefore$ $20x^2-x-12$ has rational factors. Find the factors.

242. Surd Factors of a Trinomial. When we say that a trinomial can be factored, we usually mean that it can be expressed as the product of rational factors.

As we have seen, this can always be done when b^2-4ac is a perfect square.

When there are no rational roots we may use the preceding method to find **surd factors**.

Ex.—Find two surd factors of x^2-6x+4.

If $x^2-6x+4=0$, $x=\dfrac{6\pm\sqrt{20}}{2}=3\pm\sqrt{5}$.

$$\therefore\quad x^2-6x+4=(x-3-\sqrt{5})(x-3+\sqrt{5}).$$

Verify by multiplication.

EXERCISE 155

Factor, by trial if you can, otherwise by solving the corresponding equations and verify:

1. $3x^2-17x+10$.

2. $20x^2+3x-108$.

3. $x^2-2x-1763$.

4. $1800a^2-5a-1$.

5. $299x^2+10x-1$.

6. $221x^2-458ax+221a^2$.

7. Show that $12x^2-15x+4$ has no rational factors.

8.* x^2+4x-3 has no rational factors. Find two surd factors of it.

9. If x^2-8x+k is a perfect square, find k.

10. If $ax^2-kx+9a$ is a perfect square, when the factor a is removed, find k.

11. Express $x^2-6x-11$ as the product of two surd factors.

12. Factor $144x^4-337x^2y^2+144y^4$. When this expression is equal to zero, find four values of the ratio of x to y.

13. If $x-2$ is a factor of $120x^3-167x^2-ax+56$, find the value of a and find the other two factors.

14. By finding the square root of ax^2+bx+c, find the relation which must connect a, b and c when this expression is a perfect square.

243. A Quadratic Equation cannot have more than two Roots. We have seen that the equation $ax^2+bx+c=0$ has two roots, and since this equation represents every quadratic, it follows that every quadratic equation has two roots.

It cannot have more than two roots.

Let m and n be the roots of $ax^2+bx+c=0$.

Then $$ax^2+bx+c=a(x-m)(x-n) \quad \text{(art. 240).}$$

$$\therefore \quad a(x-m)(x-n)=0.$$

Since this product is zero, one factor must be zero. But a is not zero, for the equation would not then be a quadratic. Therefore, either

$$x-m=0 \text{ or } x-n=0.$$

But no values of x other than m and n will make either of these quantities equal to zero.

$$\therefore \quad m \text{ and } n \text{ are the only roots.}$$

Since the quadratic equation $ax^2+bx+c=0$ has only two roots, then *the quadratic expression* ax^2+bx+c *can be resolved into linear factors in only one way.*

EXERCISE 157 (Review of Chapter XXV)

1. What is the sum and the product of the roots of $ax^2+bx+c=0$?

2. Under what condition are the roots of $ax^2+bx+c=0$ reciprocals? When are they equal in magnitude but opposite in sign ?

3. When are the roots of $ax^2+bx+c=0$, (1) equal, (2) real, (3) imaginary, (4) rational ?

4.* If $p+q=4$ and $pq=5$, find the values of :

$$p^2+q^2,\quad \frac{1}{p^2}+\frac{1}{q^2},\quad \frac{p}{q}+\frac{q}{p},\quad p^3+q^3,\quad p^4+q^4.$$

5. Find the sum and the product of the roots of

$$(3x-2)(x-3)=(x-1)(x-5).$$

6. Find the equation whose roots are each one-half of the roots of $4x^2-20x+21=0$.

7. Find the sum of the squares of the roots of $3x^2-11x+1=0$.

8. Find the equation whose roots are twice as great as the roots of $24x^2-38x+15=0$.

9. For what value of k will $x^2-10x=k$ have equal roots ?

10. Find the equation whose roots are m and n when $m^2+n^2=74$ and $mn=35$.

11. Factor $5256x^2+x-1$ and $221x^2-8x-165$.

12. Form the equation whose roots are m and n where $m^3+n^3=28$ and $m+n=4$.

13. Find the sum of the roots of $(x-a)^2+(x-b)^2=(x-c)^2$.

14. Construct the equation whose roots are the reciprocals of the roots of $17x^2+53x-97=0$.

15. Express x^2+6x+7 as the product of two linear factors.

16. Construct the equation whose roots are each greater by 7 than the roots of $2x^2+11x-21=0$.

17. Find the equation whose roots are each three times the roots of $ax^2+bx+c=0$.

18. If m and n are the roots of $ax^2+bx+c=0$, find the equation whose roots are $\frac{m}{n}$ and $\frac{n}{m}$.

19. Show that $(a+b+c)x^2-2x(a+b)+a+b-c=0$ has rational roots. What are they?

20. If one root of $x^2-px+q=0$ is double of the other, show that $2p^2=9q$.

21. If m and n are the roots of $x^2+px+q=0$, show that p and q are the roots of $x^2+x(m+n-mn)-mn(m+n)=0$.

22. Show that the equation $\left(mx+\frac{a}{m}\right)^2=4ax$ has equal roots for all values of m.

23. Find the values of k for which the equation

$$x^2+x(2+k)+k+37=0$$

has equal roots.

24. Since $x^2-8x-20=(x-10)(x+2)$, for what values of x is the expression $x^2-8x-20$ equal to zero? For what values is it negative? For what values is it positive?

25. Show that it is impossible to divide a line 6 inches in length into two parts such that the area of the rectangle contained by them may be 10 square inches.

26. For what values of k is $4x^2-x(k+8)+k+5$ a perfect square? Verify your result.

27. Find the sum of the cubes of the roots of $x^2+mx+n=0$.

28. Find the sum of the squares of the roots of

$$x^2-x(1+a)+\tfrac{1}{2}(1+a+a^2)=0.$$

29. Find the sum and the product of the roots of

$$\frac{a}{x-a}+\frac{b}{x-b}=\frac{c}{x-c}.$$

30. If the sum of the roots of $ax^2-6x+12a=0$ equals their product, find a and verify.

31. It is evident that a is one root of $(x-c)(x-b)=(a-c)(a-b)$. Find the other root.

32. If $x^2-5x-3a$ and $x^2-11x+3a$ have a common factor, it must be a factor of their difference. Make use of this to find the value of a for which $x^2-5x-3a=0$ and $x^2-11x+3a=0$ will have a common root. Verify by finding the roots.

33. The absolute term in an equation of the form $x^2+px+q=0$ is misprinted 18 instead of 8. A student in consequence finds the roots to be 3 and 6. What were the roots meant to be?

34. If m and n are the roots of $ax^2+bx+c=0$, show that $m+n$ and $\frac{1}{m}+\frac{1}{n}$ are the roots of $acx^2+bx(a+c)+b^2=0$.

35. Two boys attempt to solve a quadratic equation. After reducing it to the form $x^2+px+q=0$, one of them has a mistake only in the absolute term and finds the roots to be 1 and 7. The other has a mistake only in the coefficient of x, and finds the roots to be -1 and -12. What were the correct roots?

36. Express $x^2+2bx+c$ as the product of two linear factors in x.

CHAPTER XXVI

SUPPLEMENTARY THEOREMS AND EXERCISES

ADDITIONAL EXAMPLES IN FACTORING

244. Product of two Trinomials. If we multiply

$$a-2b-3c \text{ by } 2a-b+c$$

the product may be written

$$2a^2-5ab+2b^2-5ac+bc-3c^2.$$

The first three terms of the product, which do not contain the letter c, are evidently the product of $a-2b$ and $2a-b$.

The last term, $-3c^2$, is the product of $-3c$ and c.

If we wish to factor the product of two trinomials, we may do so by the method of cross multiplication, which we used to factor a trinomial.

Ex. 1.—Factor $2a^2-5ab+2b^2-5ac+bc-3c^2$.

First factor $2a^2-5ab+2b^2$, and then choose such factors of $-3c^2$ as will give the remaining terms in the product when the complete multiplication is performed:

$$\begin{array}{llllll} 2a^2 & -5ab & +2b^2 & -5ac & +bc & -3c^2 \\ a & & -2b & & & -3c \\ 2a & & -b & & & +c. \end{array}$$

If the terms of the factors are written under the terms from which they are obtained, it is not difficult to obtain by trial the factors of an expression of this type.

Ex. 2.—Factor $4a^2+3b^2-12c^2-8ab-8ac$.

Arrange the expression thus:

$$\begin{array}{lllll} 4a^2 & -8ab & +3b^2 & -8ac & -12c^2 \\ 2a & & -b & & +2c \\ 2a & & -3b & & -6c \end{array}$$

Show by multiplication that these factors are correct.

EXERCISE 136

Write, mentally, the products of :

1. $\begin{array}{l} a-2b+c \\ \underline{a-\ b-c} \end{array}$

2. $\begin{array}{l} 3x-y+1 \\ \underline{3x+y-2} \end{array}$

3. $\begin{array}{l} 3a-4b+c \\ \underline{2a-\ b-2c} \end{array}$

4. $\begin{array}{l} 3x+\ y-4 \\ \underline{2x-3y+3} \end{array}$

5. $\begin{array}{l} a-b+4 \\ \underline{2a-b} \end{array}$

6. $\begin{array}{l} 2x-5y+\ z \\ \underline{3x-2y-3z} \end{array}$

7. $\begin{array}{l} 2a-3b-5c \\ \underline{2a+3b} \end{array}$

8. $\begin{array}{l} 3m-2n+1 \\ \underline{2m-3n+4} \end{array}$

9. $\begin{array}{l} a^2-2a+3 \\ \underline{a^2+3a-2} \end{array}$

Factor and verify :

10. $a^2-4ab+4b^2-a+2b-12.$

11. $2x^2+xy-6y^2+2xz+11yz-4z^2.$

12. $2a^2+6b^2-3c^2+7bc-5ca-7ab.$

13. $2x^2-7xy-22y^2-5x+35y-3.$

14. $6a^2+ab-12b^2-2a+31b-20.$

15. $x^2-xz-6z^2-2xy+6yz.$

16.* Divide the product of $6a^2-5ab+b^2+11a-4b+3$ and $a+b-2$ by $3a^2+2ab-b^2-5a+3b-2.$

17. Reduce to lowest terms

$$\frac{4p^2+21q^2-18r^2+33qr+6rp-31pq}{4p^2-7pq+3q^2-2pr+3qr-6r^2}.$$

18. If $3x+2y-5z$ is a factor of

$$3x^2+axy-6y^2+bxz+cyz-10z^2,$$

what are the values of a, b and c ?

19. Write the expression $x^2+xy-2y^2-x+10y-12$ in the form $x^2+x(y-1)-(2y^2-10y+12)$. Solve the corresponding equation for x and thus find the factors of the given expression.

20. Solve $x^2-5ax+6a^2+7x-17a+12=0$. (1) by factoring. (2) by the general formula.

21. Express, in the factor form, the L.C.M. of

$$6a^2-5ab+b^2+ac-c^2$$

and $$6a^2+ab-2b^2-ac+4bc-2c^2.$$

245. Sum and Difference of Cubes. We have seen that

$$a^3+b^3=(a+b)(a^2-ab+b^2), \quad a^3-b^3=(a-b)(a^2+ab+b^2),$$
$$(a+b)^3=a^3+3a^2b+3ab^2+b^3, \quad (a-b)^3=a^3-3a^2b+3ab^2-b^3.$$

Similarly, $(a+b)^3+c^3=(a+b+c)\{(a+b)^2-c(a+b)+c^2\}$,

and $(a-b)^3-c^3=(a-b-c)\{(a-b)^2+c(a-b)+c^2\}$.

Ex. 1.—Factor $a^3+b^3+c^3-3abc$.

Add to a^3+b^3 sufficient to make the sum the cube of $a+b$, that is, add $3a^2b+3ab^2$.

Then
$$\begin{aligned} & a^3+b^3+c^3-3abc, \\ &=a^3+b^3+3a^2b+3ab^2+c^3-3a^2b-3ab^2-3abc, \\ &=(a+b)^3+c^3-3ab(a+b+c), \\ &=(a+b+c)\{(a+b)^2-c(a+b)+c^2-3ab\}, \\ &=(a+b+c)(a^2+b^2+c^2-ab-bc-ca). \end{aligned}$$

The factors of this expression are important, and the pupil should endeavour to retain them in memory.

The expression is the sum of the cubes of three quantities diminished by three times their product.

One factor is the sum of the three quantities, and the other is the sum of their squares **diminished** *by the sum of their products taken two at a time.*

We should recognize expressions which are of the same form as this type expression.

Thus, $a^3+b^3-c^3+3abc$ may be written in the form

$$a^3+b^3+(-c)^3-3ab(-c),$$

and it is now seen to be the sum of the cubes of a, b and $-c$, diminished by three times their product.

The factors of $a^3+b^3-c^3+3abc$ may at once be written down from the factors of the type form by merely substituting $-c$ for c.

$$\therefore \quad a^3+b^3-c^3+3abc=(a+b-c)(a^2+b^2+c^2-ab+bc+ca).$$

$$\begin{aligned} a^3-b^3-c^3-3abc&=a^3+(-b)^3+(-c)^3-3a(-b)(-c), \\ &=(a-b-c)(a^2+b^2+c^2+ab-bc+ca). \end{aligned}$$

$$\begin{aligned} 8x^3+27y^3-z^3+18xyz&=(2x)^3+(3y)^3+(-z)^3-3(2x)(3y)(-z), \\ &=(2x+3y-z)(4x^2+9y^2+z^2-6xy+3yz+2zx \end{aligned}$$

Ex. 2.—Factor $a^3+b^3+1-3ab$.

$$a^3+b^3+1-3ab=a^3+b^3+1^3-3ab\,.\,1.$$
$$=(a+b+1)(a^2+b^2+1-ab-a-b).$$

Ex. 3.—Find one factor of

$$(x+y)^3+(y+z)^3+(z+x)^3-3(x+y)(y+z)(z+x).$$

This is of the same form as $a^3+b^3+c^3-3abc$, where $a=x+y$, $b=y+z$, $c=z+x$.

One factor is $x+y+y+z+z+x$ or $2(x+y+z)$.

The other factor is lengthy, but is easily written down.

Ex. 4.—If $a+b+c=0$, show that $a^3+b^3+c^3=3abc$.

This is equivalent to showing that $a^3+b^3+c^3-3abc=0$.

Now this quantity will be equal to zero, if one of its factors is zero. But $a+b+c$ is already seen to be a factor, and since it is given equal to zero,

$$\therefore\quad a^3+b^3+c^3-3abc=0, \text{ or } a^3+b^3+c^3=3abc.$$

We have thus shown that *if the sum of three quantities is zero, the sum of their cubes is equal to three times their product.*

Prove this also by substituting $-b-c$ for a.

Ex. 5.—Show that

$$(a-b)^3+(b-c)^3+(c-a)^3=3(a-b)(b-c)(c-a).$$

Here the sum of $a-b$, $b-c$, $c-a$ is zero, and therefore the result follows at once from the preceding theorem.

Similarly, $(a+2b-3c)^3+(b+2c-3a)^3+(c+2a-3b)^3$
$$=3(a+2b-3c)(b+2c-3a)(c+2a-3b),$$
since the sum of $a+2b-3c$, $b+2c-3a$, $c+2a-3b$ is zero.

EXERCISE 169

Factor:

1. $(a+2b)^3-c^3$.
2. $a^3-(b-c)^3$.
3. $(a+b)^3+8c^3$.
4. $(a+b)^3+(c+d)^3$.
5. $(x-y)^3-(a-b)^3$.
6. $(2x-y)^3+(x-2y)^3$.
7. $(3a-b)^3-(a-3b)^3$.
8. $8(3a-b)^3-27(2a-3b)^3$.
9. $a^3-b^3+c^3+3abc$.
10. $8x^3+y^3+z^3-6xyz$.

11. $a^3+b^3-1+3ab$. **12.** $1+c^3-d^3+3cd$.

13.* $8x^3-y^3-125z^3-30xyz$. **14.** $(a+b)^3+c^3+1-3c(a+b)$.

What is the product of:

15. $a-b-c$ and $a^2+b^2+c^2+ab+ac-bc$.

16. $2x-y+3z$ and $4x^2+y^2+9z^2+2xy-6xz+3yz$.

17. $1-a-b$ and $1+a^2+b^2+a+b-ab$.

18. $2a-3b-4$ and $4a^2+6ab+9b^2-12b+8a+16$.

What is the quotient of:

19. $1-a^3+b^3+3ab$ by $1-a+b$.

20. $27m^3-n^3-1-9mn$ by $3m-n-1$.

21. $a^3+125b^3-1+15ab$ by $a^2+25b^2+1-5ab+a+5b$.

What is one factor of:

22. $(4a+3b)^3-(a+2b)^3$.

23. $(x^2-3x+7)^3+8$.

24. $(a^2-3a+2)^3-(a^2-5a+7)^3$.

25. $(a+b)^3+(c+d)^3-1+3(a+b)(c+d)$.

26. Prove that the difference of the cubes of $4a^2+a+1$ and $2a^2-2a+3$ is divisible by the product of $2a-1$ and $a+2$.

27. Show that $a^3+b^3+c^3-3abc$ is equal to

$$\tfrac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}.$$

28. Write down a quantity of the type $a^3+b^3+c^3-3abc$, of which $3x-2y+z$ is a factor. What is the other factor?

29. If $a+b-c=0$, show that $a^3+b^3+3abc=c^3$.

30. If $x=y+z$, show that $x^3=y^3+z^3+3xyz$

31. If $a+b+c=0$, show that

$$(a+2b)^3+(b+2c)^3+(c+2a)^3=3(a+2b)(b+2c)(c+2a).$$

32. Show that

$$(x-y)^3+(y-z)^3+(z-x)^3-3(x-y)(y-z)(z-x)=0.$$

33. Show that $(a+3b-4c)^3+(b+3c-4a)^3+(c+3a-4b)^3$

$$=3(a+3b-4c)(b+3c-4a)(c+3a-4b).$$

34. If $x=a-b$, $y=a+b$, $z=2a$, show that $x^3+y^3+3xyz=z^3$.

35. Find the value of $a^3-b^3+c^3+3abc$ when $a=\cdot32$, $b=\cdot46$, $c=\cdot14$.

36. Reduce to lowest terms:

$$\frac{a^3-b^3-c^3-3abc}{2a^2-5ab+3b^2+ac-3c^2} \text{ and } \frac{x^3+8y^3+z^3-6xyz}{(x-2y)^2+(2y-z)^2+(z-x)^2}.$$

37. Find two factors of the first degree of

$$(ax+by+az)^3+(bx+ay+bz)^3.$$

38. When $x=b+c$, $y=c+a$, $z=a+b$, prove that

$$x^3+y^3+z^3-3xyz=2(a^3+b^3+c^3-3abc).$$

39. Prove that $a^2+b^2+c^2-ab-ac-bc$ is unaltered if a, b, c be each increased, or each decreased by the same quantity.

40. Solve $(x-a)^3+(b-x)^3+(a-b)^3=0$.

246. Grouping Terms. We have already seen (art. 91) that we can frequently obtain a factor of an expression by a suitable arrangement of the terms.

The following examples will give further illustrations of this method.

Ex. 1.—Factor $a^2(b-c)+b^2(c-a)+c^2(a-b)$.

Arrange in descending powers of a, and the expression

$$\begin{aligned}
&=a^2(b-c)-a(b^2-c^2)+bc(b-c),\\
&=(b-c)(a^2-ab-ac+bc),\\
&=(b-c)\{a(a-b)-c(a-b)\},\\
&=(b-c)(a-b)(a-c).
\end{aligned}$$

When the factors are written in cyclic order (art. 140),

$$a^2(b-c)+b^2(c-a)+c^2(a-b)=-(a-b)(b-c)(c-a).$$

This expression may also be factored by writing it in the equivalent form $(a^2-b^2)(b-c)-(a-b)(b^2-c^2)$.

In this form $a-b$ and $b-c$ are seen to be factors. Complete the factoring by this method.

The expression $a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)$ differs only in sign from $a^2(b-c)+b^2(c-a)+c^2(a-b)$,

$$\therefore \quad a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)=(a-b)(b-c)(c-a).$$

Also, $\quad ab(a-b)+bc(b-c)+ca(c-a)=-(a-b)(b-c)(c-a)$.

Ex. 2.—Factor $a^3(b-c)+b^3(c-a)+c^3(a-b)$.

The expression $= a^3(b-c)-a(b^3-c^3)+bc(b^2-c^2)$,

$= (b-c)(a^3-ab^2-abc-ac^2+b^2c+bc^2)$.

Now arrange the second factor in powers of b, and proceed as before and obtain $-(a-b)(b-c)(c-a)(a+b+c)$.

Factor also by using the second method of Ex. 1, writing the expression in the form $(a^3-b^3)(b-c)-(a-b)(b^3-c^3)$.

What are the factors of $a(b^3-c^3)+b(c^3-a^3)+c(a^3-b^3)$, and of $ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)$?

Ex. 3.—Factor $a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$.

Arrange in descending powers of a, and the expression

$$= a^2(b+c)+a(b^2+2bc+c^2)+bc(b+c),$$
$$= (b+c)(a^2+ab+ac+bc),$$
$$= (b+c)(a+b)(a+c) = (a+b)(b+c)(c+a).$$

Ex. 4.—Factor $(a^2-b^2)x^2+(a^2+b^2)x+ab$.

Expressions of this kind, when written in descending powers of x, are easily factored by cross multiplication in the usual way.

$$\begin{array}{ll} (a^2-b^2)x^2+(a^2+b^2)x+ab & \\ (a\ +b\ )x & +a \\ (a\ -b\ )x & +b \end{array}$$

The factors are $(a+b)x+a$ and $(a-b)x+b$.

EXERCISE 160

Factor and verify 1-8 :

1. $acx^2+x(ad+bc)+bd$.
2. $mpx^2+xy(qm-pn)-nqy^2$.
3. $x^2(a^2-b^2)+4abx-(a^2-b^2)$.
4. $(p^2-q^2)y^2+2y(p^2+q^2)+p^2-q^2$.
5. $x^2(a+b)+x(a+2b+c)+b+c$.
6. $x^2(a^2-a)+x(2a^2-3a+2)+a^2-2a$.
7. $a^2(b+c)+a(b^2+3bc+c^2)+bc(b+c)$.
8. $ab(a+b)+bc(b+c)+ca(c+a)+3abc$

9.* $x^2(y-z)+y^2(z-x)+z^2(x-y)$.

10. $xy(x-y)+yz(y-z)+zx(z-x)$.

11. $x(y^2-z^2)+y(z^2-x^2)+z(x^2-y^2)$.

12. $a(b^3-c^3)+b(c^3-a^3)+c(a^3-b^3)$.

13. $a^2(b^4-c^4)+b^2(c^4-a^4)+c^2(a^4-b^4)$.

14. Divide

$$a^2(b-c)+b^2(c-a)+c^2(a-b) \text{ by } a^2(b-c)+b^2(c-a)+c^2(a-b).$$

Solve and verify:

15. $abx^2-x(ad+bc)+cd=0$.

16. $(a^2-b^2)x^2-4abx=a^2-b^2$.

17. $x^2(a-b)+a^2(b-x)+b^2(x-a)=0$.

18. $abx^2-x(a^2+b^2)+a^2-b^2=0$.

19. $(a^2-ab)x^2+(a^2+b^2)x=ab+b^2$.

20. Find a common factor of

$$abx^2+x(a^2-2ab-b^2)-a^2+b^2 \text{ and } a^2x^2-a^2x-ab-b^2.$$

247. The Factor Theorem. We have already seen that any expression is divisible by $x-a$, if the expression vanishes when we substitute a for x (art. 101).

Any expression whose value depends on the value of x is called a function of x (art. 114).

Any function of x may be conveniently represented by the symbol $f(x)$, which is read "function x."

The factor theorem might be stated thus:

$$f(x) \text{ is divisible by } x-a \text{ if } f(a)=0.$$

Thus, if $f(x)=x^3-7x^2+11x-2$,

$f(2)=8-28+22-2=0$

$\therefore$ $x^3-7x^2+11x-2$ is divisible by $x-2$.

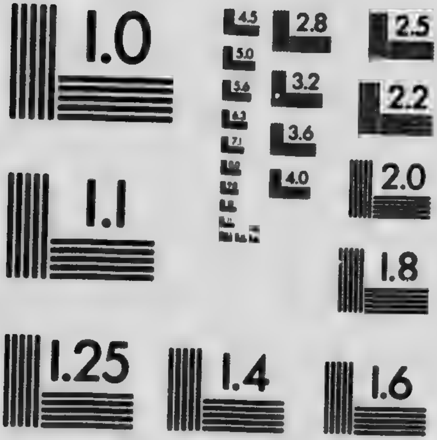
If $f(x)=x^3-4x^2a+5xa^2+10a^3$,

then $f(-a)=-a^3-4a^3-5a^3+10a^3=0$.

$\therefore$ $x^3-4x^2a+5xa^2+10a^3$ is divisible by $x+a$.

MICROCOPY RESOLUTION TEST CHART

(ANSI and ISO TEST CHART No. 2)



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248. Factors of $x^n \pm a^n$. We have already seen that

$$x^2-a^2=(x-a)(x+a),$$
$$x^3-a^3=(x-a)(x^2+xa+a^2),$$
$$x^4-a^4=(x^2-a^2)(x^2+a^2)=(x-a)(x+a)(x^2+a^2).$$

Here we see that $x-a$ is a factor of each.

Is $x-a$ a factor of x^5-a^5 ?

When we substitute a for x,

$$x^5-a^5=a^5-a^5=0.$$

$\therefore$ $x-a$ is a factor of x^5-a^5.

(1) Is $x-a$ a factor of x^n-a^n ?

When we substitute a for x,

$$x^n-a^n=a^n-a^n=0.$$

$\therefore$ x^n-a^n is divisible by $x-a$.

(2) Is $x+a$ a factor of x^n-a^n ?

When we substitute $-a$ for x,

$$x^n-a^n=(-a)^n-a^n.$$

Now $(-a)^n-a^n$ will be equal to zero only when $(-a)^n=a^n$, and this is true only when n is even,

$\therefore$ x^n-a^n is divisible by $x+a$ when n is even.

Thus, x^2-a^2, x^4-a^4, x^6-a^6, etc., are divisible by $x+a$, but x^3-a^3, x^5-a^5, etc., are not divisible by $x+a$.

(3) Is $x+a$ a factor of x^n+a^n ?

Examine this, as in the preceding, and show it is a factor only when n is odd.

(4) Is $x-a$ a factor of x^n+a^n ?

We thus conclude that, when n is a positive integer,

(1) $\mathrm{x}^n-\mathrm{a}^n$ *is always divisible by* $\mathrm{x}-\mathrm{a}$.

(2) $\mathrm{x}^n-\mathrm{a}^n$ *is divisible by* $\mathrm{x}+\mathrm{a}$ *when* n *is even.*

(3) $\mathrm{x}^n+\mathrm{a}^n$ *is divisible by* $\mathrm{x}+\mathrm{a}$ *when* n *is odd.*

(4) $\mathrm{x}^n+\mathrm{a}^n$ *is never divisible by* $\mathrm{x}-\mathrm{a}$.

249. Quotient on dividing $x^n \pm a^n$ by $x \pm a$

(1) $\dfrac{x^2-a^2}{x-a}=x+a.$ $\qquad$ $\dfrac{x^4-a^4}{x-a}=x^3+x^2a+xa^2+a^3.$

$\dfrac{x^3-a^3}{x-a}=x^2+xa+a^2$ $\qquad$ $\dfrac{x^5-a^5}{x-a}=x^4+x^3a+x^2a^2+xa^3+a^4$

Verify these results by division or multiplication.

Notice that the signs are all positive, and that the powers of x are descending and those of a are ascending.

Similarly, $\dfrac{x^7-a^7}{x-a}=x^6+x^5a+x^4a^2+x^3a^3+x^2a^4+xa^5+a^6,$

and $\dfrac{x^n-a^n}{x-a}=x^{n-1}+x^{n-2}a+x^{n-3}a^2+\ldots+xa^{n-2}+a^{n-1}.$

(2) $\dfrac{x^2-a^2}{x+a}=x-a.$ $\qquad\dfrac{x^4-a^4}{x+a}=x^3-x^2a+xa^2-a^3.$

Verify and note the peculiarity in the signs.

Write down the value of $\dfrac{x^6-a^6}{x+a}$ and of $\dfrac{x^{2n}-a^{2n}}{x+a}$.

(3) $\dfrac{x^3+a^3}{x+a}=x^2-xa+a^2.$ $\qquad\dfrac{x^5+a^5}{x+a}=x^4-x^3a+x^2a^2-xa^3+a^4.$

Write down the value of $\dfrac{x^7+a^7}{x+a}$ and of $\dfrac{x^{2n+1}+a^{2n+1}}{x+a}$.

EXERCISE 161

1. If $f(x)=x^3-8x^2+19x-12$, find the values of

$f(1),\ f(2),\ f(3),\ f(4),\ f(5).$

What are the factors of $x^3-8x^2+19x-12$?

2. If $f(x)=x^4-2x^3-x^2+2x$, find the values of

$f(2),\ f(1),\ f(0),\ f(-1),\ f(-2).$

What are the factors of $f(x)$ in this case ?

3. Prove that $x^{14}-y^{14}$ is divisible by $x-y$ and $x+y$.

4. Prove that $x^{12}-1$ is divisible by $x-1$, $x+1$, x^2+1, x^3+1.

5. Prove that $x^{17}+y^{17}$ is divisible by $x+y$ and that x^5+32 is divisible by $x+2$.

Write down the quotients in the following divisions :

6. $\dfrac{x^3+y^3}{x+y}$. 7. $\dfrac{a^4-b^4}{a-b}$. 8. $\dfrac{a^4-b^4}{a+b}$. 9. $\dfrac{x^7-1}{x-1}$.

10.* $\dfrac{x^5+32}{x+2}$. 11. $\dfrac{x^4-81}{x+3}$. 12. $\dfrac{x^{10}-a^5}{x^2-a}$. 13. $\dfrac{(a+b)^4-1}{(a+b)+1}$.

14. State one factor of:

$$x^3-b^3,\ a^7+b^7,\ x^6-64,\ m^3+\frac{1}{m^3},\ (x+y)^5-1.$$

What is the product of:

15. a^3+a^2+a+1 and $a-1$.

16. $m^4-m^3+m^2-m+1$ and $m+1$.

17. $a^6+a^4b^2+a^2b^4+b^6$ and a^2-b^2.

18. Prove that $x^6+3x^4+4x^2+224$ is divisible by x^2+7.

19. Show that $x+y$, x^2+y^2, x^3+y^3, x^4+y^4, x^6+y^6 and $x^{12}+y^{12}$ are factors of $x^{24}-y^{24}$.

20. If $x-a$ is a factor of x^2+px+q, find the relation between a, p and q.

21. If $f(x)=mx^2+nx+r$, find $f(a)$ and show that $f(x)-f(a)$ is divisible by $x-a$.

22. If $x-1$ is a factor of $x^3-k^2x^2+10kx-10$, find the values of k and verify.

23. Write down the quotient when

(1) $x-a$ is divided by $x^{\frac{1}{3}}-a^{\frac{1}{3}}$.

(2) $x+a$ is divided by $x^{\frac{1}{3}}+a^{\frac{1}{3}}$.

(3) $x-a$ is divided by $x^{\frac{1}{5}}-a^{\frac{1}{5}}$.

(4) $x+a$ is divided by $x^{\frac{1}{5}}+a^{\frac{1}{5}}$.

250. Symmetrical Expressions. An expression is said to be **symmetrical** with respect to any two letters if it is unaltered when those two letters are interchanged.

Thus, $x+y$ and x^2+y^2 are symmetrical with respect to x and y, but x^2+xy is not symmetrical.

Similarly, $a+b+c$ and $ab+bc+ca$ are symmetrical with respect to a and b, b and c, c and a, for if any two be interchanged the expressions remain unaltered.

251. Cyclic Symmetry. An expression is said to be symmetrical with respect to the letters a, b and c, if it is unaltered when a is changed to b, b to c and c to a, that is, when the letters are taken in **cyclic order**.

Thus, $a^2+b^2+c^2-ab-bc-ca$ is symmetrical with respect to a, b and c, for when the letters are changed in cyclic order the result is

$$b^2+c^2+a^2-bc-ca-ab,$$

which is equal to the given expression.

The expression $a^3+b^3+c^3-3abcd$ is symmetrical with respect to a, b and c, but not with respect to a, b, c and d.

The only expression of the first degree which is symmetrical with respect to a, b and c is $a+b+c$ or some multiple of it as $k(a+b+c)$.

There are two expressions of the second degree, $a^2+b^2+c^2$ and $ab+bc+ca$, and the sum of any multiples of these, such as

$$k(a^2+b^2+c^2)+l(ab+bc+ca),$$

which are symmetrical with respect to a, b, c.

252. Symmetry applied to Factoring. The factor theorem may be applied to the factoring of many symmetrical expressions.

Ex. 1.—Factor $a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)$.

If we put $a=b$, the expression equals zero,

$$\therefore \quad a-b \text{ is a factor.}$$

Since the expression is symmetrical and $a-b$ is shown to be a factor, it follows that $b-c$ and $c-a$ must be factors.

We have thus found three factors each of the first degree. But the given expression is of the third degree, and, therefore, there cannot be another literal factor. There may be a numerical factor.

Suppose k is a numerical factor,

$$\therefore \quad a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)=k(a-b)(b-c)(c-a).$$

Since this relation is true for all values of a, b, c,

let $$a=1,\ b=2,\ c=0,$$

then $$1(4-0)+2(0-1)+0=k(1-2)(2-0)(0-1),$$

$$\therefore \quad 2=2k, \text{ or } k=1,$$

$$\therefore \quad a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)=(a-b)(b-c)(c-a).$$

In finding the value of k, any values of a, b, c may be used provided they do not make both sides of the identity vanish on substitution.

Ex. 2.—Factor

$$(a+b+c)^3+(a-b-c)^3+(b-c-a)^3+(c-a-b)^3.$$

If we put $a=0$, the expression vanishes,

$\therefore$ a must be a factor, and, therefore, b and c.

Complete the solution as before, and show that the expression equals $24abc$.

Ex. 3.—Factor $a^3(b-c)+b^3(c-a)+c^3(a-b)$.

As in Ex. 1, show that $a-b$, $b-c$, $c-a$ are factors.

Since the expression is of the fourth degree it must have another factor of the first degree.

The remaining factor must be of the form $k(a+b+c)$.

$$\therefore \quad a^3(b-c)+b^3(c-a)+c^3(a-b)=k(a-b)(b-c)(c-a)(a+b+c).$$

Substitute numerical values for a, b and c and show that the factors are

$$-(a-b)(b-c)(c-a)(a+b+c).$$

Ex. 4.—Simplify

$$(a-b-2c)^2+(b-c-2a)^2+(c-a-2b)^2+(a+b+c)^2.$$

This expression is symmetrical with respect to a, b and c and is of the second degree.

In the simplified result there can be only two kinds of terms, squares like a^2 and products like ab.

The coefficient of a^2 in the result is $1+4+1+1$ or 7,

$\therefore$ one part of the result is $7(a^2+b^2+c^2)$.

The coefficient of ab is $-2-4+4+2=0$,

$\therefore$ the complete result is $7(a^2+b^2+c^2)$.

Check by letting $a=b=c=1$.

Ex. 5.—Simplify

$$(a+b)(a+b-2c)+(b+c)(b+c-2a)+(c+a)(c+a-2b).$$

The coefficient of a^2 in the result is $1+1$ or 2,

$\therefore$ one part of the result is $2(a^2+b^2+c^2)$.

The coefficient of ab is $2-2-2$ or -2,

$\therefore$ the other part of the result is $-2(ab+bc+ca)$,

$\therefore$ the complete result is $2(a^2+b^2+c^2-ab-bc-ca)$.

EXERCISE 162 (1-12, Oral)

With respect to what letters are these symmetrical:

1. $a+b$.
2. $a+c-b$.
3. x^2+y^2+xy.
4. $ab+bc+ca$.
5. $a^3+b^3+c^3-3abc$.
6. x^2+y^2+x-y.
7. $3(p^2+q^2+r^2)-2(pq+qr+rp)$.

8. What is the simplest expression of the first degree which is symmetrical with respect to x and y? a, b and c? a, b, c and d?

9. What expression similar to a^2+b^2+3ab is symmetrical with respect to a, b and c?

10. Simplify

$$(a+b)^2+(b+c)^2+(c+a)^2 \quad\text{and}\quad (a-b)^2+(b-c)^2+(c-a)^2.$$

11. If $a+b$ is a factor of any expression, symmetrical with respect to a, b and c, what other factors must it have?

12. When $(a+b)^3+(b+c)^3+(c+a)^3$ is simplified, the coefficient of a^3 is 2, of a^2b is 3 and of abc is 0. What must the simplified form be?

Simplify:

13.* $(a-b+c)^2+(b-c+a)^2+(c-a+b)^2$.

14. $(a+b)(a+b-c)+(b+c)(b+c-a)+(c+a)(c+a-b)$.

15. $(x-y)(px+py-z)+(y-z)(py+pz-x)+(z-x)(pz+px-y)$.

16. $(a-b)^3+(b-c)^3+(c-a)^3$.

Factor:

17. $x^2(y-z)+y^2(z-x)+z^2(x-y)$

18. $xy(x-y)+yz(y-z)+zx(z-x)$.

19. $a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$.

20. $(a+b+c)^3-(a+b-c)^3-(b+c-a)^3-(c+a-b)^3$.

21. $(x-y)^3+(y-z)^3+(z-x)^3$.

22. $a(b+c)^2+b(c+a)^2+c(a+b)^2-4abc$.

23. $ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)$.

24. $a^2(b^4-c^4)+b^2(c^4-a^4)+c^2(a^4-b^4)$.

Simplify:

25. $$\frac{x(y+z)}{(x-y)(z-x)}+\frac{y(z+x)}{(y-z)(x-y)}+\frac{z(x+y).}{(z-x)(y-z)}.$$

26. $$\frac{x^2}{(x-y)(x-z)}+\frac{y^2}{(y-z)(y-x)}+\frac{z^2}{(z-x)(z-y)}.$$

27. $$\frac{ab}{(c-a)(c-b)}+\frac{bc}{(a-b)(a-c)}+\frac{ca}{(b-c)(b-a)}.$$

28. $$\frac{a}{bc(a-b)(c-a)}+\frac{b}{ca(b-c)(a-b)}+\frac{c}{ab(c-a)(b-c)}.$$

29. $$\frac{b^2-ac}{(a-b)(b-c)}+\frac{c^2-ba}{(b-c)(c-a)}+\frac{a^2-cb}{(c-a)(a-b)}.$$

30. $$\frac{bc(b+c)}{(a-b)(a-c)}+\frac{ca(c+a)}{(b-c)(b-a)}+\frac{ab(a+b)}{(c-a)(c-b)}.$$

31. $$\frac{x^3}{(x-y)(z-x)}+\frac{y^3}{(y-z)(x-y)}+\frac{z^3}{(z-x)(y-z)}.$$

32. $$\frac{(a-b)^3+(b-c)^3+(c-a)^3}{a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)}.$$

33. Simplify $(a+b+c)^3-(b+c)^3-(c+a)^3-(a+b)^3+a^3+b^3+c^3$, being given that a is a factor of it.

34. Show that $a-b$ is a factor of

$$a^n(b-c)+b^n(c-a)+c^n(a-b).$$

What may be inferred regarding other factors?

35. An expression is symmetrical in x, y and z and each term is of two dimensions. When $x=y=z=1$, the expression equals 15, and when $x=1$, $y=2$, $z=3$, it equals 64. Find the expression.

36. Point out wherein it is obviously impossible for the following statements to be true:

(1) $(a^2+b^2+c^2)(a+b+c)=a^3+b^3+a^2(b+c)+b^2(c+a)$.

(2) $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-3ab)$.

(3) $(a-b)(b-c)(c-a)=ab^2+b^2c+ca^2-ac^2-bc^2-ba^2$.

253. Identities. We have already had many examples of algebraic expressions which are **identically equal**, that is, which are equal for all values of the letters involved.

Thus,
$$(x+y)(x-y)=x^2-y^2,$$
$$(x+y+z)^2=x^2+y^2+z^2+2xy+2xz+2yz,$$
$$(a+b)^3=a^3+3a^2b+3ab^2+b^3,$$
$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca).$$

Any of these may be shown to be identities by performing the operations necessary to remove the brackets on one side, when the result is the same as the other side.

Ex.—Show that $(a+b+c)^3$

$$=a^3+b^3+c^3-3abc+3(a+b+c)(ab+bc+ca).$$

Here the cube of $a+b+c$ may be found by multiplication or by any other method.

The brackets are then removed from the right and the terms collected.

The two sides are now the same, which shows that the given statement is an identity.

We might also have changed the second side into the first by factoring, thus :

$$(a^3+b^3+c^3-3abc)+3(a+b+c)(ab+bc+ca),$$
$$=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)+3(a+b+c)(ab+bc+ca),$$
$$=(a+b+c)(a^2+b^2+c^2+2ab+2ac+2bc),$$
$=(a+b+c)^3$, which proves the proposition.

254. When two expressions are to be shown equal, the result may frequently be obtained by showing that their difference is zero.

The difference may be zero,

(1) *because all of the terms cancel, or*

(2) *because it has a factor which is equal to zero, identically, or which is given equal to zero.*

Ex. 1.—Prove

$$(a-b)^3+(b-c)^3+(c-a)^3=3(a-b)(b-c)(c-a).$$

Here we may prove that

$$(a-b)^3+(b-c)^3+(c-a)^3-3(a-b)(b-c)(c-a)=0,$$

(1) by removing the brackets when all the terms cancel,

(2) by observing that $(a-b)+(b-c)+(c-a)$ is a factor of the expression and this factor is identically equal to zero (art. 245).

Ex. 2.—If $a+b=c$, show that $a^2+bc=b^2+ca$.

Here, as in the preceding, we may show that $a^2+bc-b^2-ca=0$, by showing that $a+b-c$ is a factor of it and this factor is given equal to zero, or by substituting $c=a+b$ in each side or in the difference. Solve this problem both ways.

Ex. 3.—If $a+b+c=0$, show that

$$(a+b)(b+c)(c+a)+abc=0.$$

For $a+b$ substitute $-c$, for $b+c$ substitute $-a$, and for $c+a$ substitute $-b$ and

$$(a+b)(b+c)(c+a)+abc=(-c)(-a)(-b)+abc=0.$$

Ex. 4.—If $2s=a+b+c$, prove that

$$s^2+(s-a)^2+(s-b)^2+(s-c)^2=a^2+b^2+c^2.$$

When the first side is simplified it

$$=4s^2-2s(a+b+c)+a^2+b^2+c^2,$$
$$=4s^2-2s(2s)+a^2+b^2+c^2,$$
$$=a^2+b^2+c^2, \text{ which was required.}$$

Of course, this could have been proven by substituting the value of s at once. It is usually easier, however, to substitute in the last step.

EXERCISE 163

Prove the following identities:

1. $a(b+c)^2+b(c+a)^2+c(a+b)^2-4abc=(a+b)(b+c)(c+a)$.
2. $(x+y)^4+x^4+y^4=2(x^2+xy+y^2)^2$.
3. $(a+b)^3+(a-b)^3+6a(a+b)(a-b)=8a^3$.

4. $2(a^3+b^3+c^3-3abc)=(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}$.

5. $a(b-c)^3+b(c-a)^3+c(a-b)^3=(a-b)(b-c)(c-a)(a+b+c)$.

If $a+b+c=0$, show that:

6. $(3a-2b+4c)^2-(2a-3b+3c)^2=0$.

7. $a^2+b^2-c^2+2ab=0$ and $c^2-ab=b^2-ac$.

8. $(a+b)(b+c)+(b+c)(c+a)+(c+a)(a+b)=ab+bc+ca$.

9. $a^4+b^4+c^4=2a^2b^2+2b^2c^2+2c^2a^2$.

10. $(3a-b)^3+(3b-c)^3+(3c-a)^3=3(3a-b)(3b-c)(3c-a)$.

11. $a(b^2+bc+c^2)+b(c^2+ca+a^2)+c(a^2+ab+b^2)=0$.

12. If $a+b=1$, prove that $(a^2-b^2)^2=a^3+b^3-ab$.

13. If $x+y=2z$, prove that $\dfrac{x}{x-z}+\dfrac{y}{y-z}=2$.

14. If $a=\dfrac{y-z}{x}$, $b=\dfrac{z-x}{y}$, $c=\dfrac{x-y}{z}$, show that $a+b+c+abc=0$.

15. If $\dfrac{1}{a}+\dfrac{1}{a-c}=\dfrac{2}{a-b}$, prove that $\dfrac{1}{a}+\dfrac{1}{b}=\dfrac{2}{c}$.

16. If $x+\dfrac{1}{x}=y$, show that $x^2+\dfrac{1}{x^2}=y^2-2$ $x^3+\dfrac{1}{x^3}=y^3-3y$; $x^4+\dfrac{1}{x^4}=y^4-4y^2+2$.

If $2s=a+b+c$, show that:

17. $s(s-a)+(s-b)(s-c)=bc$.

18. $a(s-a)+b(s-b)+c(s-c)+2s^2=2(ab+bc+ca)$.

19. $(s-a)^2+(s-b)^2+(s-c)^2+2(s-a)(s-b)+2(s-b)(s-c)$
$+2(s-c)(s-a)=s^2$.

20. $(2as+bc)(2bs+ca)(2cs+ab)=(a+b)^2(b+c)^2(c+a)^2$.

21. $\dfrac{1}{s-a}+\dfrac{1}{s-b}+\dfrac{1}{s-c}-\dfrac{1}{s}=\dfrac{abc}{s(s-a)(s-b)(s-c)}$.

22. $16s(s-a)(s-b)(s-c)=2b^2c^2+2c^2a^2+2c^2a^2-a^4-b^4-c^4$.

23. If $b+\dfrac{1}{c}=1$, $c+\dfrac{1}{a}=1$, prove $a+\dfrac{1}{b}=1$ and $abc=-1$.

24.* If $a+\frac{1}{a}=3$, find the value of $a^3+\frac{1}{a^3}$.

25. If $a=x(b+c)$, $b=y(c+a)$, $c=z(a+b)$, show that

$$xy+yz+zx+2xyz=1.$$

26. If $x+y=a$ and $xy=b^2$, find the values of x^2+y^2 and x^3+y^3 in terms of a and b.

27. Eliminate x and y from the equations $x+y=a$, $xy=b^2$, $x^2+y^2=c^2$.

28. Eliminate x and y from $x+y=a$, $x^2+y^2=b^2$, $x^3+y^3=c^3$.

29. If $x=a+b-c$, $y=b+c-a$, $z=c+a-b$, show that

$$x^3+y^3+z^3-3xyz=4(a^3+b^3+c^3-3abc).$$

EXERCISE 164 (Review of Chapter XXVI)

1. Show that $x^3+y^3+z^3-3xyz$ is divisible by $x+y+z$, and hence show that $(b-c)^3+(c-a)^3+(a-b)^3=3(a-b)(b-c)(c-a)$.

2. Prove that

$$\left(a^2+b^2+c^2\right)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)-\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{c}{a}+\frac{a}{c}\right)\left(\frac{a}{b}+\frac{b}{a}\right)=1.$$

3. If $a+b+c+d=0$, prove that

$$(a+b)(a+c)(a+d)=(b+c)(b+d)(b+a).$$

4. Prove that $(a-b)^n+(b-c)^n+(c-a)^n$ is divisible by

$$(a-b)(b-c)(c-a),$$

when n is an odd integer.

5. If n is a positive integer prove that 12^n-1 is divisible by 11, $23^{2n+1}+1$ by 24, $7^{2n}-1$ by 48.

6.* Write down a quantity of the same type as $x^3+y^3+z^3-3xyz$ of which $\frac{1}{2}x+\frac{1}{3}y-\frac{1}{4}z$ is a factor.

7. Show that a, $a-x$ and $a-2x$ are factors of

$$(a-b)(a-b-x)(a+2b-2x)+b(b-x)(3a-2b-2x).$$

8. Show that $(x+y)^n-x^n-y^n$ is always divisible by $xy(x+y)$, when n is an odd integer.

9. If $(y-a)(1-a)=(y-b)(1-b)=x$, find x in terms of a and b only.

10. If $x+y+z=0$, prove that

(1) $x^2+xy+y^2=y^2+yz+z^2=z^2+zx+x^2$.

(2) $(x+y-z)^3+(y+z-x)^3+(z+x-y)^3+24xyz=0$.

11. Simplify $\dfrac{a}{bc(a-b)(a-c)}+\dfrac{b}{ca(b-c)(b-a)}+\dfrac{c}{ab(c-a)(c-b)}$.

12. Solve $(x-a)^3+(x-b)^3+(x-c)^3=3(x-a)(x-b)(x-c)$.

13. Show that $(a+b)^5-a^5-b^5=5ab(a+b)(a^2+ab+b^2)$.

14. If $2s=a+b+c$, show that

(1) $s(s-b)+(s-a)(s-c)=ac$.

(2) $s^2+(s-a)(s-b)+(s-b)(s-c)+(s-c)(s-a)=ab+bc+ca$.

(3) $(s-a)^2+(s-c)^2+3b(s-a)(s-c)=b^2$.

15. Prove that $a^n(b^2-c^2)+b^n(c^2-a^2)+c^n(a^2-b^2)$ is divisible by $(a-b)(b-c)(c-a)$ and find the quotient when $n=3$.

16. Simplify $\dfrac{x-a}{a(a-b)(a-c)}+\dfrac{x-b}{b(b-c)(b-a)}+\dfrac{x-c}{c(c-a)(c-b)}$.

17. If $x=a^2-bc$, $y=b^2-ca$, $z=c^2-ab$, prove that

$$ax+by+cz=(a+b+c)(x+y+z).$$

18. Simplify $\dfrac{a^3(b-c)+b^3(c-a)+c^3(a-b)}{(b-c)^3+(c-a)^3+(a-b)^3}$.

19. If $ab+bc+ca=0$, show that

(1) $(a+b+c)^2=a^2+b^2+c^2$.

(2) $(a+b+c)^3=a^3+b^3+c^3-3abc$.

(3) $(a+b+c)^4=a^4+b^4+c^4-4abc(a+b+c)$.

20. Show that $x^{n+1}-x^n-x+1$ is divisible by $(x-1)^2$, when n is a positive integer.

21. Write down the quotient on dividing

x^4-a^4 by $x-a$, x^6+1 by x^2+1, a^5-32 by $a-2$.

22. Factor $x^4-1-3(x^3-1)+4(x^2-1)$.

23. Simplify $\dfrac{a(b^2+bc+c^2)}{(a-b)(a-c)}$ + two similar fractions.

24. Show that $x(y^2-z^2)+y(z^2-x^2)+z(x^2-y^2)$ is not altered when x is changed to $x+a$, y to $y+a$, z to $z+a$.

25. If $x^2=x+1$, show that $x^5=5x+3$.

26. Find two linear factors of

$$(ax+b)^3+(bx+c)^3+(cx+a)^3-3(ax+b)(bx+c)(cx+a).$$

27. If $x^{\frac{2}{3}}+y^{\frac{2}{3}}=z^{\frac{2}{3}}$, show that $(x^2+y^2-z^2)^3+27x^2y^2z^2=0.$

28. If $a+b+c=0$, prove that

$$a^3+b^3+c^3+3(a+b)(b+c)(c+a)=0.$$

29. A homogeneous expression of two dimensions is symmetrical in x, y, z. Its value is 42 when $x=y=z=2$ and is 16 when $x=1$, $y=2$, $z=0$. Find it.

30. Eliminate x and y from $x+y=a$, $xy=b$, $x^2+y^2=c$.

31. If $x+y=3$ and $x^2+y^2=5$, find the values of x^3+y^3 and x^4+y^4.

32. If $a+b+c=10$ and $ab+bc+ca=31$, find the values of $a^2+b^2+c^2$ and $a^3+b^3+c^3-3abc$.

CHAPTER XXVII

PROGRESSIONS

255. Series. A sequence of quantities in which each is formed from one or more of the preceding by a definite law is called a **series.** The quantities are called the **terms** of the series.

Consider the following series :

(1) 2, 5, 8, 11, ... (2) 13, 11, 9, 7, ...
(3) 3, 6, 12, 24, . . . (4) 81, 27, 9, 3, ...

What is the law of formation in each of these? What would be the next four terms in each?

ARITHMETIC PROGRESSION

256. A series in which each term after the first is formed by adding a constant quantity to the preceding term is called an **arithmetic series or progression (A.P.)**

The constant quantity is called the **common difference** and is found by subtracting any term from the term which *follows* it.

The following are examples of arithmetic series:

(1) 1, 3, 5, 7, ... (2) 15, 5, —5, —15, ...
(3) 7½, 0, —7½, ... (4) a, $a+d$, $a+2d$, ...

The common difference in (1) is 2, in (2) is —10, in (3) is —7½ and in (4) is d.

257. General Form of an A.P. If we let a be the first term and d the common difference, the general form of an A.P. is

$$\mathbf{a,\ a+d,\ a+2d,\ a+3d, \ldots\ldots}$$

Since a and d may have any values, this series represents every arithmetic series.

The $3rd$ term $= a+2d$, the $4th$ term $= a+3d$,
,, $10th$,, $= a+9d$, ,, $37th$,, $= a+36d$.

That is, the coefficient of d in any term is one less than the number of the term.

$\therefore$ **the nth term $= a+(n-1)\,d$.**

Any term of an A.P. may be found by substituting in this formula the particular values of a, d and n.

Ex. 1.—Find the 7th, 20th and nth terms of

(1) $2, 7, 12, 17, \ldots.$ (2) $15, 12\frac{1}{2}, 10, \ldots$

In (1) $a=2$, $d=5$, $\therefore$ the 7th term $= 2+6\times5 = 32$,
,, 20th ,, $= 2+19\times5 = 97$,
,, nth ,, $= 2+(n-1)5 = 5n-3$.
In (2) $a=15$, $d=-2\frac{1}{2}$, $\therefore$,, 7th ,, $= 15+6\times(-2\frac{1}{2}) = 0$,
,, 20th ,, $= 15-19\times2\frac{1}{2} = -32\frac{1}{2}$,
,, nth ,, $= 15-(n-1)2\frac{1}{2} = 17\frac{1}{2}-2\frac{1}{2}n$.

Ex. 2.—Which term of $8, 11, 14, \ldots.$ is 68?

Suppose 68 is the nth term, then since $a=8$, $d=3$,

$$68 = 8+(n-1)3 \text{ or } n = 21.$$

$\therefore$ 68 is the 21st term.

Verify by finding the 21st term.

Ex. 3.—Determine the A.P. in which the 3rd term is 10 and the 14th term is 54, and find the nth term.

Let the series be a, $a+d$, $a+2d, \ldots..$
$\therefore$ $a+2d = 10$ and $a+13d = 54$.
Solving we find that $d = 4$ and $a = 2$.
$\therefore$ the series is $2, 6, 10, 14, \ldots..$
The nth term is $2+(n-1)4 = 4n-2$

EXERCISE 165

Find the 5th and 12th terms of the following series:

1.* $3, 8, 13, \ldots$ 2. $11, 6, 1, \ldots\ldots$ 3. $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\ldots$

4. $3\frac{1}{2}, 2, \frac{1}{2}, \ldots$ 5. $a, 3a, 5a \ldots$ 6. $1{\cdot}2, {\cdot}8, {\cdot}4, \ldots$

7. $a+b, a, a-b, \ldots\ldots\ldots$ 8. $2a-b, 3a-2b, 4a-3b, \ldots$

9. $x+1, 2x+3, 3x+5, \ldots$ 10. $m+n, 0, -m-n, \ldots\ldots$

11. Find the 100th term of $1, 3, 5, \ldots\ldots$ and the 50th term of $7\frac{3}{4}, 9\frac{1}{2}, 11\frac{1}{4}, \ldots\ldots$

12. Find the 20th term of $\frac{1}{\sqrt{2}}, \sqrt{2}, \frac{3}{\sqrt{2}}, \ldots\ldots$ and the 23rd term of $\frac{3}{\sqrt{5}}, \frac{4}{\sqrt{5}}, \sqrt{5}, \ldots\ldots$

Find the nth term of:

13. $1, 2, 3, \ldots$ 14. $1, 3, 5, \ldots$ 15. $7, 10, 13, \ldots$

16. $2, -1, -4, \ldots$ 17. $3x, x, -x, \ldots$ 18. $n-1, n-2, n-3, \ldots$

19. $(a+b)^2, a^2+b^2, (a-b)^2, \ldots$ 20. $2-\frac{1}{n}, 2-\frac{2}{n}, 2-\frac{3}{n}, \ldots$

Which term of the series

21. $1, 3, 5, \ldots$ is 39? 22. $9, 6, 3, \ldots$ is -288?

23. $21, 19\frac{1}{2}, 18, \ldots$ is 0? 24. $\frac{3}{4}, \frac{1}{2}, \frac{1}{4}, \ldots$ is -3?

25. If the first term of an A.P. is 3 and the 20th term is 79, what is the common difference?

26. If the 8th term of an A.P. is 50 and the 21st term is 115, find the 50th term.

27. If the first two terms of an A.P. are 2 and 5 and the last term is 119, find the number of terms.

28. Find the first term of an A.P. in which the 15th term is -40 and the 31st term is 40.

29. If the 3rd term of an A.P. is 9 and the 10th term is -12, which term is -60?

30. How many odd numbers are there from 13 to 193 both inclusive?

31. How many multiples of 7 are between 100 and 500?

32. The sum of the 2nd and 5th terms of an A.P. is 26 and of the 4th and 20th is 94. Find a and d.

33. The sum of the first three terms of an A.P. is less than the sum of the next three by 54, find d.

34. In the series 3, 7, 11,...., 199, find the 20th term from the end. (Reading from the end $a=199$, $d=-4$.)

35. Find the sum of the 8th term from the beginning and the 8th term from the end of 62, 59,.... ...., 2.

36. In the series a, $a+d$,........, $l-d$, l, show that the sum of the rth term from the beginning and the rth term from the end is constant for all values of r.

37. The sum of three numbers in A.P. is 30. Find the middle number.

38. If the sum of three numbers in A.P. is 15 and the sum of their squares is 83, find the numbers. (Let the numbers be $a-d$, a, $a+d$.)

39. If the 10th, 20th and nth terms of an A.P. are 29, 59 and 137 respectively, find n.

40. If $x+y$, $2x-2y$, $5x-6y$ are in A.P., show that $y=2x$.

41. Show that if the sides of a right angled triangle are in A.P. they are in the ratio 3 : 4 : 5. If the hypotenuse is 25, what are the sides?

42. Show that the $(n-r+1)$th, $(n+1)$th and the $(n+r+1)$th terms of any A.P. are in A.P.

258. Sum of an A.P. Suppose it is required to find the sum (S) of 8 terms of the series 3, 8, 13,...

$$S=3+8+13+18+23+28+33+38,$$
$$\text{or } S=38+33+28+23+18+13+8+3.$$

Adding on each side we get

$$\begin{aligned}2S&=41+41+41+41+41+41+41+41,\\&=41\times8,\end{aligned}$$
$$\therefore\ S=(41\times8)\div2=164.$$

Similarly if S is the sum of n terms of

a, $a+d$, $a+2d$,........ and if l denotes the last term

then $S=a+(a+d)+(a+2d)+\ldots\ldots+(l-2d)+(l-d)+l$,
or $S=l+(l-d)+(l-2d)+\ldots\ldots+(a+2d)+(a+d)+a$.
$\therefore\ 2S=(a+l)+(a+l)+(a+l)+\ldots\ldots$ to n terms,
$=n(a+l)$.

$$\therefore\ \mathbf{S=\frac{n}{2}(a+l)},\ \text{but } l=\mathbf{a+(n-1)d},$$

$$\therefore\ \mathbf{S=\frac{n}{2}\left\{2a+(n-1)d\right\}}.$$

Ex. 1.—Find the sum of 50 terms of $3+5+7+\ldots\ldots$

Here $a=3$, $d=2$, $n=50$,

$$\therefore\ S=\frac{n}{2}\left\{2a+(n-1)d\right\}=\frac{50}{2}(6+49\times 2)=2600.$$

Ex. 2.—Find the sum of all multiples of 3 between 100 and 500.

Here the first term is 102 and the last is 498.
If we suppose n to be the number of terms,
$\therefore$ the nth term $=102+(n-1)3=498$,

$$\therefore\ n=133.$$

$$\therefore\ S=\frac{n}{2}(a+l)=\frac{133}{2}(102+498)=39900.$$

Ex. 3.—How many terms of the series
$21+18+15+\ldots\ldots$ will give the sum 81?

Here $a=21$, $d=-3$, $S=81$,

$$\therefore\ 81=\frac{n}{2}\left\{42-(n-1)3\right\},$$

$$\therefore\ n^2-15n+54=0,$$

$$\therefore\ n=6 \text{ or } 9.$$

Therefore the number of terms is either 6 or 9.

Write down 9 terms of the series and discover why the sum of 6 terms is the same as the sum of 9 terms.

Ex. 4.—How many terms of

$12+16+20+\ldots$ will make 208?

Here $208=\dfrac{n}{2}\left\{24+(n-1)4\right\}$.

Solving we get $n=8$ or -13.

Therefore the number of terms is 8, the other value of n giving a solution which is inadmissible.

Ex. **5.**—The sum of n terms of an A.P. is $3n^2+2n$ for all values of n. Find the series.

If $n=1$, $3n^2+2n=5$, therefore the 1st term is 5.

If $n=2$, $3n^2+2n=16$, therefore the sum of two terms is 16.

$\therefore$ the second term is 11 and the common difference is 6.

$\therefore$ the series is $5+11+17+\ldots$.

Verify by finding the sum of n terms of this series.

EXERCISE 166

Find the sum of the arithmetic series whose first and last terms and number of terms respectively are:

1.* 1, 79; 40. **2.** 5, 75; 10. **3.** 25, -50; 32.
4. a, $25a$; 13. **5.** $-3x$, $7x$; n. **6.** $a+b$, $a-b$; $2n$.

Find the sum of:

7. 1, 4, 7, ... to 20 terms. **8.** 9, 13, 17, ... to 12 terms.
9. 12, 6, 0, „ 32 „ **10.** 2, $3\frac{1}{2}$, 5, .. „ 18 „
11. -11, $-9\frac{1}{2}$, -8, ... to 15 terms. **12.** 99, 97, 95, ... „ 99 „
13. $\frac{1}{6}$, $\frac{1}{3}$, $\frac{1}{2}$, .. to 16 terms. **14.** 1, $\frac{1}{3}$, $-\frac{1}{3}$, ... „ 18 „
15. a, $3a$, $5a$, .. to 12 terms. **16.** 1, 1·1, 1·2, .. „ 100 „

Find the sum of n terms of:

17. $2+5+8+\ldots$ **18.** $1+2+3+\ldots$ **19.** $5+11+17+\ldots$
20. $3+1-1-\ldots$ **21.** $4+1{\cdot}5-1-\ldots$ **22.** $a+5a+9a+\ldots$
23. $(a+1)+(a+3)+(a+5)+\ldots$
24. $\left(1-\dfrac{1}{n}\right)+\left(1-\dfrac{2}{n}\right)+\left(1-\dfrac{3}{n}\right)+\ldots$

25. Find the sum of the first n odd numbers.

26. Find the sum of 20 consecutive odd numbers of which the greatest is 125.

27. If the sum of $16+14+12+\ldots$ is 72, find the number of terms.

28. If $a=4\frac{1}{2}$, $d=-\frac{1}{4}$, and $S=41\frac{1}{4}$, find n.

29. If $3+3\frac{1}{8}+3\frac{1}{4}+\ldots$ to n terms $=68$, find n.

30. If the 5th term of an A.P. is 8 and the 10th term is -2, find the sum of 17 terms.

31. If the sum of the 4th and 8th terms of an A.P. is 24, and of the 15th and 19th is 68, find the sum of n terms.

32. Find the A.P. in which the sum of the first 10 terms is 100 and the sum of the next 10 is 300.

33. A travels 2 miles the first hour, $2\frac{1}{4}$ the second, $2\frac{1}{2}$ the third and so on, B travels 4 miles per hour. In how many hours will A overtake B, if they start together?

34. There are 40 stones in a row, 1 yard apart. How far does a boy travel in bringing them together, one by one, at the first stone?

35. Show that the sum of $2n+1$ terms of an A.P. is $2n+1$ times the middle term.

36. Find the sum of all multiples of 6 between 1000 and 2000.

37. Find the sum of all the integers from 10 to 100, both included, which are not divisible by 3.

38. A body falling freely falls 16·1 feet in the first second, and in each succeeding second 32·2 feet more than in the preceding second. If a stone dropped from a stationary balloon reaches the ground in 12 seconds, how high is the balloon?

39. The sum of n terms of $1+4+7+\ldots.$ is to the sum of $2n$ terms as 10:41. Find n.

40. If the second term of an A.P. is 28, the sixth is 12 and the last is -28, find the sum of all the terms.

41. With which term must I begin in the series 99, 97, 95,..... so that the sum of 40 consecutive terms will be 960?

42. If the sum of n terms of an A.P. is $2n^2-3n$, find the series.

259. Arithmetic mean. When three numbers are in A.P., the middle one is called the **arithmetic mean** between the other two.

Thus 7 is the arithmetic mean between 3 and 11 since 3, 7, 11 are in A.P.

If x is the arithmetic mean between a and b then a. x. b, are in A.P.

$$\therefore\quad x-a=b-x,$$

$$\therefore\quad x=\frac{a+b}{2}.$$

Therefore the arithmetic mean between any two quantities is one-half of their sum.

260. When any number of terms are in A.P., the terms intermediate between any two are called the **arithmetic means** between those two terms.

Thus 3, 5, 7 are three arithmetic means between 1 and 9 since 1, 3, 5, 7, 9 are in A.P.

Ex. **1.**—Insert 4 arithmetic means between 5 and 85.

The whole series consists of 6 terms of which the first is 5 and the 6th is 85.

$$\therefore\quad 85=5+5d, \text{ or } d=16,$$

$$\therefore\quad \text{the means are } 21,\ 37,\ 53,\ 69.$$

Ex. **2.**—Insert n arithmetic means between a and b.

The whole series consists of $n+2$ terms of which the first is a and the last or $(n+2)$th is b.

Therefore if d is the common difference,

$$a+(n+1)d=b, \text{ or } d=\frac{b-a}{n+1}.$$

$\therefore$ the required means are

$$a+\frac{b-a}{n+1},\quad a+\frac{2(b-a)}{n+1},\quad a+\frac{3(b-a)}{n+1},\ldots\ ,\ a+\frac{n(b-a)}{n+1}.$$

EXERCISE 167

1. Find the arithmetic mean between 16 and 20; -11 and -41; $a+b$ and $a-b$; $5x-3y$ and $3x+5y$.

2.* Insert 6 arithmetic means between 11 and 53; 8 between 5 and 11; 10 between 13 and 29½; 12 between $4a+3b$ and $3b-9a$.

3. Insert 20 arithmetic means between 2 and 65 and find their sum.

4. If 100 arithmetic means be inserted between -12 and 493, find the 65th mean.

5. When m arithmetic means are inserted between x and y, find the rth mean.

6. For what values of x is the arithmetic mean between ax^2+bx and $ax+bx^2$ equal to $a+b$?

7. If a, b, c are in A.P. and if x is the arithmetic mean between a and b, and y between b and c, show that a, x, b, y, c are in A.P.

8. If a, x, y, b are in A.P., find x and y in terms of a and b.

HARMONIC PROGRESSION

261. A sequence of quantities is said to be in **harmonic progression,** when their reciprocals are in arithmetic progression.

Thus $1, \frac{1}{2}, \frac{1}{3}, \ldots.$ are in H.P. since $1, 2, 3, \ldots$ are in A.P.
Also $15, 12, 10, \ldots$,, ,, ,, ,, $\frac{1}{15}, \frac{1}{12}, \frac{1}{10}, \ldots$,, ,, ,,

262. General Form of an H.P. Since the general form of an A.P. is $a, a+d, a+2d, a+3d, \ldots\ldots.., a+(n-1)d$, the general form of an H.P. is

$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \frac{1}{a+3d}, \ldots\ldots.., \frac{1}{a+(n-1)d}.$$

Any term of an H.P. may be found by finding the same term of the corresponding A.P. and taking its reciprocal. There is no general formula for the sum of an H.P.

Ex. **1.**—Find the 6th term and the nth term of an H.P. whose first two terms are 20 and 15.

The first two terms of the corresponding A.P. are $\frac{1}{20}, \frac{1}{15}$,

$\therefore$ the common difference of the A.P. $= \frac{1}{15}-\frac{1}{20} = \frac{1}{60}$,

$\therefore$ the 6th term of the A.P. $= \frac{1}{20}+\frac{5}{60} = \frac{2}{15}$,

and ,, nth ,, ,, ,, ,, $= \frac{1}{20}+(n-1)\frac{1}{60} = \dfrac{n+2}{60}$.

$\therefore$ the 6th term of the H.P. $= \frac{15}{2} = 7\frac{1}{2}$,

and ,, nth ,, ,, ,, ,, $= \dfrac{60}{n+2}$.

Ex. **2.**—The 12th term of an H.P. is $\frac{1}{5}$ and the 19th term is $\frac{3}{22}$, find the 4th term.

Let a be the first term and d the common difference of the corresponding A.P.

Then $a+11d=5$ and $a+18d=\frac{22}{3}$.
Solving we find that $a=\frac{4}{3}$ and $d=\frac{1}{3}$.
$\therefore$ the 4th term of the A.P. $=a+3d=\frac{7}{3}$,
$\therefore$ the 4th term of the H.P. $=\frac{3}{7}$.

263. Harmonic Mean. When three quantities are in H.P., the middle one is called the **harmonic mean** between the other two.

Let x be the harmonic mean between a and b,

$\therefore$ $\frac{1}{x}$ is the arithmetic mean between $\frac{1}{a}$ and $\frac{1}{b}$.

$$\therefore \frac{1}{x}=\tfrac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)=\frac{a+b}{2ab},$$

$$\therefore x=\frac{2ab}{a+b}.$$

Therefore *the harmonic mean between two quantities is twice their product divided by their sum.*

When three numbers are in H.P. they may be conveniently represented by $a, \frac{2ab}{a+b}, b$.

Ex. **1.**—Insert three harmonic means between 2 and 8.

The 1st term of the corresponding A.P. is $\frac{1}{2}$ and the 5th is $\frac{1}{8}$,

$$\therefore \tfrac{1}{8}=\tfrac{1}{2}+4d, \quad \therefore d=-\tfrac{3}{32}.$$

$\therefore$ the terms of the A.P. are $\frac{1}{2}, \frac{13}{32}, \frac{10}{32}, \frac{7}{32}, \frac{1}{8}$,
$\therefore$ the required harmonic means are $\frac{32}{13}, \frac{16}{5}, \frac{32}{7}$.

Ex. 2.—If a, b, c are in H.P., prove that $\frac{a}{c}=\frac{a-b}{b-c}$.

Since a, b, c are in H.P. $b=\frac{2ac}{a+c}$,

$$\therefore\quad \frac{a-b}{b-c}=\frac{a-\frac{2ac}{a+c}}{\frac{2ac}{a+c}-c}=\frac{a^2-ac}{ac-c^2}=\frac{a(a-c)}{c(a-c)}=\frac{a}{c}.$$

EXERCISE 168

Show that the following are in H.P. and find the next two terms in each:

1.* 60, 30, 20. **2.** $\frac{3}{4}$, 1, $\frac{3}{2}$. **3.** 4, 2, $\frac{4}{3}$.

4. 4·8, 6, 8. **5.** 6, —12, —3. **6.** $\frac{1}{\sqrt{2}}$, $\sqrt{2}-1$, $\frac{2-\sqrt{2}}{2}$.

7. Find the 24th term of the H.P., 24, 12, 8,....

8. Find the nth term of the H.P., $\frac{1}{2}$, $\frac{4}{7}$, $\frac{2}{3}$,....

9. Find the nth term of an H.P. whose first term is a and second term is b.

10. Find the sum of 6 terms in H.P. if the first term is 60 and the second is 30.

11. The 5th term of an H.P. is 168 and the 8th is 105, find the first two terms.

12. The first term of an H.P. is 2·4 and the 4th is —3, find which term is —·3.

13. What is the harmonic mean between 30 and 15; —3 and 4, $\frac{1}{2}$ and $\frac{1}{3}$; $\frac{a-b}{a+b}$ and $\frac{a+b}{a-b}$?

14. Insert 3 harmonic means between 1 and 2; 4 between —7 and $\frac{1}{3}$; 10 between 3 and $\frac{1}{4}$.

15. If a, b, c are in H.P. prove that $a:a-b=a+c:a-c$.

16. The sum of three numbers in H.P. is 37. If the middle one is 12, find the others.

17. Find two numbers whose difference is 8 and such that the harmonic mean between them is 15.

18. If a, b, c are in H.P. prove that $\frac{b+a}{b-a}+\frac{b+c}{b-c}=2$.

19. If a, x, y, b are in H.P. find x and y in terms of a and b.

20. What number must be added to each of the numbers 1, 3, 9 so that the sums will be in H.P.?

21. Find x so that $a+x$, $b+x$, $c+x$ may be in H.P.

GEOMETRIC PROGRESSION

264. A series in which each term after the first is formed by multiplying the preceding term by a constant quantity is called a **geometric progression** (G.P.).

The constant quantity is called the **common ratio** of the series and is found by dividing any term into the term which *follows* it.

The following are examples of geometric series:

(1) $1, 2, 4, 8, \ldots\ldots$ (2) $6, 3, \frac{3}{2}, \frac{3}{4}, \ldots\ldots$

(3) $2, -6, 18, -54, \ldots\ldots$ (4) $a, ar, ar^2, ar^3, \ldots\ldots$

The common ratio in (1) is 2, in (2) is $\frac{1}{2}$, in (3) is -3 and in (4) is r.

265. General Form of a G.P. If we let a be the first term and r the common ratio, the general form of a G.P. is

$$\mathbf{a, ar, ar^2, ar^3, \ldots\ldots}$$

Since a and r may have any values, this series represents every geometric series.

The 4th term is ar^3, the 7th is ar^6, the 20th is ar^{19}.

That is, the index of r in each term is one less than the number of the term.

$$\therefore \textbf{ the nth term} = \mathbf{ar^{n-1}}.$$

Ex. 1.—Find the 4th, 7th and nth terms of

(1) 2, 6, 18,..... (2) 16, —8, 4,....,

In (1) $a=2$, $r=3$, $\therefore$ the 4th term $=2.3^3=54$,

$\quad$ „ 7th „ $=2.3^6=1458$,

$\quad$ „ nth „ $=2.3^{n-1}$.

In (2) $a=16$, $r=-\frac{1}{2}$, $\therefore$ „ 4th „ $=16\left(-\frac{1}{2}\right)^3=-2$,

$\quad$ „ 7th „ $=16\left(-\frac{1}{2}\right)^6=\frac{1}{4}$,

$\quad$ „ nth „ $=16\left(-\frac{1}{2}\right)^{n-1}$,

$$=2^4\frac{(-1)^{n-1}}{2^{n-1}}=\frac{(-1)^{n-1}}{2^{n-5}}.$$

The nth term in (2) is ambiguous in sign, being positive when n is odd and negative when n is even.

Ex. 2.—Which term of 243, 81, 27,.... is $\frac{1}{81}$?

Suppose $\frac{1}{81}$ is the nth term, then since $a=243$, $r=\frac{1}{3}$,

$$\therefore\ 243\left(\tfrac{1}{3}\right)^{n-1}=\tfrac{1}{81},$$

$$\therefore\ 3^{n-1}=243\times81=3^5\times3^4=3^9.$$

$$\therefore\ n-1=9 \text{ or } n=10.$$

$\therefore$ the 10th term is $\frac{1}{81}$. Verify this.

Ex. 3.—Find the G.P. in which the 6th term is 4 and the 9th term is 32.

Here $ar^5=4$ and $ar^8=32$.

$\therefore$ by division, $r^3=8$ or $r=2$,

$\therefore\ a.2^5=4$ or $a=\frac{1}{8}$.

$\therefore$ the series is $\frac{1}{8}$, $\frac{1}{4}$, $\frac{1}{2}$,....... Verify this.

266. Geometric Mean. When three numbers are in G.P., the middle one is called the **geometric mean** between the other two.

Thus, 8 is the geometric mean between 4 and 16, since 4, 8, 16 are in G.P.

If x is the geometric mean between a and b, then a, x, b are in G.P.

$$\therefore\ \frac{x}{a}=\frac{b}{x} \text{ or } x=\pm\sqrt{ab}.$$

Therefore the geometric mean between any two quantities is he square root of their product. It is usual to consider the ositive square root only.

Ex. **1.**—Insert 5 geometric means between 8 and $\frac{1}{8}$.

The whole series consists of 7 terms of which the first is 8 and the 7th is $\frac{1}{8}$.

$$\therefore \ \tfrac{1}{8} = 8r^6,$$
$$\therefore \ r^6 = \frac{1}{64} = \frac{1}{2^6},$$
$$\therefore \ r = \tfrac{1}{2}.$$

$\therefore$ the means are 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$.

Ex. **2.**—Insert n geometric means between a and b.

The whole series consists of $n+2$ terms of which the first is a and the last or $(n+2)$th is b.

$$\therefore \ ar^{n+1} = b, \ \therefore \ r = \sqrt[n+1]{\frac{b}{a}} = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}.$$

$\therefore$ the required means are

$$ar, \ ar^2, \ ar^3, \ldots\ldots \ ar^n, \text{ where } r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}.$$

EXERCISE 169

Find the required term in each of the following:

1.* 10th term of 1, 2, 4, ... **2.** 7th term of 9, 3, 1, ...

3. 6th „ „ 9, 6, 4, ... **4.** 8th „ „ 40, —20, 10, ...

5. 20th „ „ 1, x, x^2, ... **6.** 12th „ „ 32, —16, 8, ...

7. 11th „ „ a^2, ab, b^2, ... **8.** 21st „ „ a, $-a^2b$, a^3b^2, ...

9. Find the 7th and nth terms of x^2, x, 1, ...

10. Find the $(2n)$th and $(2n+1)$th terms of 1, $-\frac{1}{2}$, $\frac{1}{4}$, ...

11. Find the nth term of $(a-b)^2$, a^2-b^2, $(a+b)^2$, ...

12. If 18 is the first term of a G.P. and $\frac{2}{3}$ is the fourth term, find the common ratio.

13. If the 3rd term of a G.P. is 40 and the 5th term is 160, find the series.

14. Which term of 5, 10, 20,.... is 640?

15. The 3rd term of a G.P. is 1 and the 6th is $-\frac{1}{8}$, what is the 10th term?

16. The 4th term of a G.P. is ·016 and the 7th term is ·000128 find the first term.

17. What is the geometric mean between 4 and 9; 3 and 243; 16 and $\frac{1}{4}$; $(a-b)^2$ and $(a+b)^2$.

18. Insert two numbers between 5 and 135 so that the four numbers may form a G.P.

19. Insert 4 geometric means between 2 and 64; 5 between 3 and $7\frac{1}{3}$; 3 between x^4 and x^4y^4.

20. Find the geometric mean between the 5th and 9th terms of 3, 6, 12,....

21. The geometric mean between two numbers is 8 and the arithmetic mean is 10. Find the numbers.

22. The geometric mean between two numbers is greater than one of them by 20 and less than the other by 100. Find the numbers.

23. The sum of the 3rd and 4th terms of a G.P. is 40 and of the 6th and 7th is 2560. Find the series.

24. If a, b, c, d are in G.P. show that $ad = bc$.

25. If a, b, c are in G.P. and x is the geometric mean between a and b, and y between b and c, show that a, x, b, y, c are in G.P.

26. Show that the reciprocals of the terms of a G.P. are in G.P. Is this true concerning an A.P.?

27. Show that the series of rth terms of a G.P. are in G.P.

28. If all the terms of a G.P. be multiplied (or divided) by the same quantity, the results are in G.P.

29. The sum of three numbers in G.P. is 21 and their product is 216. Find the numbers.

30. If the arithmetic mean between two numbers is equal to the geometric mean, show that the numbers are equal.

31. Prove that the product of any two terms equidistant from the beginning and the end of a G.P. is constant.

32. Show that the geometric mean between a and b is a geometric mean between the arithmetic and harmonic means.

267. Sum of a G.P. Let S denote the sum of n terms of a series whose first term is a and common ratio r.

$$\text{Then } S = a+ar+ar^2+\ldots\ldots+ar^{n-2}+ar^{n-1}$$

$$\therefore \quad Sr = \quad ar+ar^2+\ldots\ldots+ar^{n-2}+ar^{n-1}+ar^n.$$

By subtracting, $S-Sr = a-ar^n$,

$$\therefore S(1-r) = a(1-r^n),$$

$$\therefore \mathbf{S} = \frac{\mathbf{a(1-r^n)}}{\mathbf{1-r}} \quad \textbf{or} \quad \frac{\mathbf{a(r^n-1)}}{\mathbf{r-1}}.$$

The sum of any G.P. may be found by substituting the particular values of a, r and n in this formula. If r is less than unity it is more convenient to use the first form.

Ex. **1.**—Find the sum of 10 terms of $2+4+8+\ldots\ldots$

Here $a = 2$, $r = 2$, $n = 10$,

$$\therefore S = \frac{a(r^n-1)}{r-1} = \frac{2(2^{10}-1)}{2-1} = 2046.$$

Ex. **2.**—Sum $27-9+3-1+\ldots\ldots$ to 7 terms.

Here $a = 27$, $r = -\frac{1}{3}$, $n = 7$,

$$\therefore S = \frac{a(1-r^n)}{1-r} = \frac{27\left\{1-(-\frac{1}{3})^7\right\}}{1+\frac{1}{3}} = \frac{27\left(1+\frac{1}{3^7}\right)}{\frac{4}{3}} = 20\tfrac{7}{27}.$$

268. Limiting Values. Consider the quantity r^n, where r has a certain fixed value while n may be any positive integer.

(1) It is clear that if r is less than unity, as n increases r^n decreases.

Thus if $r = \frac{1}{2}$,

when $n = 2$, $r^n = \frac{1}{4} = \cdot 25$,

" $n = 3$, $r^n = \frac{1}{8} = \cdot 125$,

" $n = 10$, $r^n = \frac{1}{1024} = \cdot 000976\ldots.$,

" $n = 20$, $r^n = \frac{1}{1024^2} = \cdot 0000009\ldots$, and so on.

Also when $r<1$, we can make r^n as small as we please by taking n sufficiently large. That is, as n becomes infinitely great r^n approaches the value 0.

The symbol ∞ is used to represent a quantity which is infinitely great or infinity, and the preceding statement is written in brief form thus :

If $r<1$, *when* $n=\infty$, *the limit of* r^n *is* 0.

(2) If $r>1$, as n increases r^n also increases, and we can make r^n as large as we please by taking n sufficiently large. *There is therefore no limit to the value of* r^n *when* $r>1$.

269. The Infinite Geometric Progression. If the number of terms of a G.P. is unlimited it is called an **infinite G.P.**

The formula for the sum of n terms of $a+ar+ar^2+\ldots$ is

$$S=\frac{a(1-r^n)}{1-r},$$

This may be written $S=\dfrac{a}{1-r}-\dfrac{ar^n}{1-r}$.

Now if $r<1$ and n is indefinitely large r^n is indefinitely small and therefore $\dfrac{ar^n}{1-r}$ is also indefinitely small.

Therefore *when* $r<1$, *the limit of the sum of the infinite series* $a+ar+ar^2+\ldots$ *is* $\dfrac{a}{1-r}$,

$$\text{or } a+ar+ar^2+\ldots\ldots \text{ to } \infty=\frac{a}{1-r}.$$

Ex. 1.—Sum $4+2+1+\ldots$ to infinity.

Since $a=4$ and $r=\frac{1}{2}$, $S=\dfrac{a}{1-r}=\dfrac{4}{1-\frac{1}{2}}=8$.

Ex. 2.—Find the limit of the sum of $2-\frac{2}{3}+\frac{2}{9}-\ldots\ldots$

Since $a=2$ and $r=-\frac{1}{3}$, $S=\dfrac{2}{1+\frac{1}{3}}=\frac{3}{2}$.

EXERCISE 170

Sum the following geometric series:

1.* $1+2+4+$..to 9 terms. **2.** $1+\frac{1}{2}+\frac{1}{4}+\ldots$ to 10 terms.

3. $\frac{1}{24}+\frac{1}{12}+\frac{1}{6}+\ldots$ to 8 terms. **4.** $16-8+4-\ldots$ to 9 terms.

5. $1+\sqrt{3}+3+\ldots$ to 8 terms. **6.** $\cdot 1+\cdot 01+\cdot 001+\ldots$ to 6 terms.

7. $1-2+4-\ldots$. to $2n$ terms. **8.** $1-\frac{1}{2}+\frac{1}{4}-\ldots$ to $(2n+1)$ terms.

9. Sum to n terms $a+2a^2+4a^3+\ldots$

10. Find the sum of $1+1\cdot 04+1\cdot 04^2+1\cdot 04^3+\ldots$.. to 15 terms, being given that $1\cdot 04^{15}=1\cdot 80094$.

11. Find the sum of the successive powers of 2 from 2 to 4096 inclusive, if $4096=2^{12}$.

12. Show that there are two geometric series in which the first term is 18 and the third term is 8 and find the sum of each to 5 terms.

13. If n be an even integer, prove that
$(1+x+x^2+\ldots+x^n)(1-x+x^2-\ldots.+x^n)=1+x^2+x^4+\ldots+x^{2n}$.

Find the limit of the sum of:

14. $16+8+4+\ldots$ **15.** $1+\frac{1}{3}+\frac{1}{9}+\ldots$ **16.** $1-\frac{1}{2}+\frac{1}{4}-\ldots$..

17. $5+\frac{10}{3}+\frac{20}{9}+\ldots$ **18.** $3+\sqrt{3}+1+\ldots$ **19.** $\cdot 5+\cdot 05+\cdot 005+\ldots$

20. $a-\frac{1}{4}a+\frac{1}{16}a-\ldots$. **21.** $5+\sqrt{15}+3+\ldots$

22. $1+1\cdot 05^{-1}+1\cdot 05^{-2}+\ldots$

23. When $r<1$, find the sum to infinity of $a-ar+ar^2-\ldots$..

24. In an infinite G.P., $S=9$ and $a=6$, find r.

25. Since $\cdot\dot{4}=\cdot 4+\cdot 04+\cdot 004+\ldots$. find a vulgar fraction which is the limit of $\cdot\dot{4}$.

26. What fraction is $\cdot 3\dot{5}\dot{4}$ or $\cdot 3+\cdot 054+\cdot 00054+\ldots$.. equal to?

27. Reduce $\cdot\dot{6}$, $\cdot\dot{1}\dot{3}$, $\cdot 01\dot{4}$ to equivalent fractions.

28. Sum to infinity $1+\dfrac{1}{1\cdot 025}+\dfrac{1}{1\cdot 025^2}+\ldots$.

29. Show that any term of the infinite series $1+\frac{1}{2}+\frac{1}{4}+$.. is equal to the sum of all the terms which follow it.

30. By how much does the sum of 12 terms of $1+\frac{1}{2}+\frac{1}{4}+\ldots$. differ from the sum to infinity?

31. Prove that the sum of the series $1+\frac{3}{5}+\frac{9}{25}+\ldots$.. and $\frac{5}{4}+\frac{5}{8}+\frac{5}{16}+\ldots$ approach each other without limit as the number of terms is increased.

32. Find the G.P. whose second term is -8 and whose sum to infinity is 18.

33. If 27, x, y, 8 are in G.P. find x and y.

34. What assumption must be made if it is possible to find the sum to infinity of $2+\frac{4x}{5}+\frac{8x^2}{25}+\dots$?

35. A ball falls to the ground from a height of 32 feet and always rises $\frac{3}{4}$ of the height from which it fell. Find the whole distance through which it will move before coming to rest.

EXERCISE 171 (Review of Chapter XXVII)

1.* Find the next two terms in each of the series:

(1) $2, 2\frac{1}{2}, 3, \dots$ (2) $2, 2\frac{1}{2}, 3\frac{1}{8}, \dots$ (3) $2, 2\frac{1}{2}, 3\frac{1}{3}$.

2. What must be added to each of the numbers 2, 8, 17 so that the results may be in G.P.?

3. If a, b, c are in A.P. prove that the following are in A.P.

(1) $a+x, b+x, c+x$. (2) $a-x, b-x, c-x$.

(3) ax, bx, cx. (4) $\frac{a}{x}, \frac{b}{x}, \frac{c}{x}$.

4. How many terms of 40, 33, 26, ... must be taken so that the sum may be 130?

5. Find three numbers in A.P. whose sum is 27 and product 693

6. Prove that the squares of x^2-2x-1, x^2+1 and x^2+2x-1 are in A.P.

7. The first term of an A.P. is 27, the fourth term is 18 and the sum of all the terms is 117, find the number of terms.

8. By how much does the sum of 7 terms of $1+\frac{1}{3}+\frac{1}{9}+\dots$ differ from the sum to infinity?

9. Find the sum of all odd integers with 3 digits.

10. If $(a^2+b^2)(b^2+c^2)=(ab+bc)^2$ then a, b, c are in G.P.

11. The A mean between two numbers is 10 and the H. mean is 6·4, find the G. mean.

12. If the nth term of an A.P. is $3n-2$, find the sum of n terms.

13. If 5, a, b, c, 29 be in A.P., find a, b, c.

14. If a, b, c, d are in G.P., prove that $a+b$, $b+c$, $c+d$ are in G.P.

15. Show that the sum of the first n odd numbers is one-third of the sum of the next n.

16. Show that $\dfrac{1+x^2}{(1-x)^2} = 1+\dfrac{2x}{1+x^2}+\dfrac{4x^2}{(1+x^2)^2}+\ldots\ldots$ to infinity.

17. The distances which a man travels on successive days are in A.P. At the end of 5 days he has gone 100 miles and at the end of 7 days 150 miles. How long will it take him to complete 300 miles in all?

18. If $x=1+a+a^2+\ldots$ to infinity and $y=1+b+b^2+\ldots$ to infinity, prove that $1+ab+a^2b^2+\ldots$ to infinity $=\dfrac{xy}{x+y-1}$.

19. The sum of a G.P. whose common ratio is 3 is 728 and the last term is 486. Find the first term.

20. A person is entitled to an annual payment which for each year is 10% less than for the year before. Show that however long he may live he cannot receive more than a certain sum in all.

21. Three quantities are in H.P. and half of the middle one is subtracted from each. Show that the remainders are in G.P.

22. If an A.P. has $2n+1$ terms, show that the sum of the first and last is double the middle term.

23. Divide $\frac{1}{12}n(n+3)$ into n parts such that each shall exceed the preceding by a fixed quantity.

24. The interior angles of a polygon are in A.P. If the least is 120° and the common difference is 5°, find the number of sides.

25. The sum of three terms in H.P. is $\frac{11}{12}$ and the first term is $\frac{1}{2}$, find the other two.

26. Find the least value of n for which the sum of the first n natural numbers exceeds 1000. Verify your result.

27. If S_n denotes the sum of n terms of an A.P., show that $S_{n+2}-2S_{n+1}+S_n=d$.

28. Sum $a+b,\ a^2+2b,\ a^3+3b,\ldots\ldots$ to n terms.

29. Find the sum of $2n$ terms of a series of which every even term is a times the term before it and every odd term is b times the term before it, the first term being 1.

CHAPTER XXVIII

PERMUTATIONS AND COMBINATIONS

270. The character of the problems to be discussed in this chapter will be illustrated by the following examples.

Ex. **1.**—How many numbers with two different digits can be formed with the digits 1, 2, 3, 4?

The numbers are 12, 13, 14; 21, 23, 24; 31, 32, 34; 41, 42, 43.

Note that there are three beginning with 1, 3 with 2, 3 with 3 and 3 with 4.

The total number $= 4 \times 3 = 12$.

Here there were 4 ways of writing down the first digit and after it was written down there were 3 ways of completing the number and the total number of ways of writing down the two digits was 4 times 3.

Ex. **2.**—In how many ways can a vowel and a consonant be selected from the 5 vowels and 21 consonants of the alphabet, the vowel being selected first?

If the vowel a is chosen, the selections are

$$ab,\ ac,\ ad,\ af,\ ag, \ldots\ldots\ldots\ ay,\ az.$$

Similarly if e is chosen, the selections are

$$eb,\ ec,\ ed,\ ef,\ eg,\ \ldots\ldots\ldots\ ey,\ ez.$$

There are 21 selections beginning with a, 21 with e, 21 with i, 21 with o and 21 with u.

$\therefore$ the total number $= 5 \times 21 = 105$.

Ex. 3.—From the letters *a*, *b*, *c*, *d*, *e* how many arrangements can be made using 2 letters? using 3 letters?

There are 5 choices for the first letter and 4 for the second.

∴ 2 letters may be written down in 5 × 4 or 20 ways.

Since each of the 3 ways of writing down the third letter may be associated with each of the 20 ways of writing down the first two letters,

∴ 3 letters may be written in 20 × 3 or 5.4.3 or 60 ways.

EXERCISE 172

1.* How many numbers with two different digits can be formed from the digits 4, 5, 6, 7, 8? Write them all down.

2. If a cent and a quarter are thrown, each falling either heads or tails, in how many ways can they fall?

3. From a baseball nine a pitcher and a catcher are to be chosen. In how many ways can it be done?

4. From a class of 20 boys and 10 girls a boy and a girl are to be chosen as leaders. In how many ways can the two be chosen?

5. A, B, C are three towns. From A to B there are three routes of travel and from B to C there are four. How many ways can I go from A to C, no two being alike?

6. In how many ways can two of the letters of the word *absurd* be written down?

7. From the letters of the alphabet, how many ways can I write down 2 different letters? 3 different letters?

8. There are 20 vacant seats in a room. In how many ways can 2 pupils be assigned seats?

9. From a committee of 10 persons a chairman, a secretary and a treasurer are to be chosen. In how many ways can this be done?

271. Permutation. Each different arrangement of some or all of a number of things is called a **permutation.**

Thus, the permutations of the letters *a*, *b*, *c*

one at a time are *a*, *b*, *c*;
two ,, ,, ,, ,, *ab*, *ac*, *ba*, *bc*, *ca*, *cb*;
three ,, ,, ,, ,, *abc*, *acb*, *bac*, *bca*, *cab*, *cba*.

The number of permutations of n things taken r at a time is represented by the symbol ${}_nP_r$.

Thus, ${}_3P_1 = 3$, ${}_3P_2 = 6$, ${}_3P_3 = 6$.

272. To find the number of permutations of n different things taken r at a time.

Let the n things be the letters $a, b, c, d, \ldots\ldots$

The first letter may be a or b or $c \ldots\ldots$, so that there are n ways of choosing the first letter.

$$\therefore \quad {}_nP_1 = n.$$

After the first letter is chosen there are $n-1$ letters left and therefore there are $n-1$ choices for the second letter. Since each choice of the second letter may be associated with each choice of the first, the number of ways of choosing 2 letters is $n(n-1)$.

$$\therefore \quad {}_nP_2 = n(n-1).$$

After 2 letters are chosen there are $n-2$ letters left and therefore there are $n-2$ choices for the third letter.

$$\therefore \quad {}_nP_3 = n(n-1)(n-2).$$

$$\text{Similarly } {}_nP_4 = n(n-1)(n-2)(n-3).$$

In each case it should be noted that the number of factors is the same as the number of things chosen.

Similarly ${}_nP_r$ would be the product of r factors, the last being $n-(r-1)$ or $n-r+1$.

$$\therefore \quad {}_n\mathbf{P}_r = n(n-1)(n-2)\ldots\ldots.(n-r+2)(n-r+1).$$

Thus, ${}_5P_2 = 5.4 = 20$, ${}_5P_3 = 5.4.3 = 60$,
${}_7P_3 = 7.6.5 = 210$, ${}_{20}P_4 = 20.19.18.17 = 116{,}280$,
${}_{2n+1}P_2 = (2n+1)2n$, ${}_{n-1}P_3 = (n-1)(n-2)(n-3)$.

$$\text{If } r = n, \text{ then } {}_nP_n = n(n-1)(n-2)\ldots\ldots.(n-n+1)$$
$$= n(n-1)(n-2)\ldots\ldots.3.2.1.$$

Therefore the number of permutations of n things all at a time, or the number of ways of arranging n things, is the product of the first n natural numbers. This product is represented by $\underline{|n}$ or $n!$ and is read "factorial n."

Thus, ${}_7P_7 = 7.6.5.4.3.2.1 = \underline{|7} = 5040$.

Ex. 1.—Find the number of ways of writing down all the letters of the word *Pictou*. (1) Where there is no restriction. (2) If the first letter must be P. (3) If the first letter must be a vowel.

(1) The result is evidently ${}_6P_6$ or $\underline{|6}$ or 720.

(2) There is only one way of choosing the first letter and the other 5 may be written in any order.

$$\therefore \text{ the number of ways} = 1 \times \underline{|5} = 120.$$

(3) There are three choices for the first letter, since there are 3 vowels. After the vowel is chosen the other 5 letters may be written in any order.

$$\therefore \text{ the number of ways} = 3 \times \underline{|5} = 360.$$

Ex. 2.—How many numbers of not more than 4 digits can be formed from the 9 digits 1, 2, ... 8, 9, no digit being repeated in any number?

The number formed with one digit is ${}_9P_1$, with two is ${}_9P_2$, with three is ${}_9P_3$ and with four is ${}_9P_4$.

$$\begin{aligned}\therefore \text{ the total number } &= {}_9P_1 + {}_9P_2 + {}_9P_3 + {}_9P_4, \\ &= 9 + 9.8 + 9.8.7 + 9.8.7.6 = 3609.\end{aligned}$$

Ex. 3.—In how many ways can 10 boys stand in a row if A and B must be together?

Since A and B must be together we may consider them as one person, but we can so consider them in two ways AB or BA.

The 9 persons could then be arranged in $\underline{|9}$ ways.

$$\therefore \text{ the result} = 2 \times \underline{|9} = 725,760.$$

EXERCISE 173

1.* Find the numerical values of ${}_{12}P_3$, ${}_7P_4$, ${}_6P_5$.

2. Show that ${}_{10}P_4 = {}_7P_7$; $\underline{|8} = 56\underline{|6}$; $n\underline{|n} = \underline{|n+1} - \underline{|n}$.

3. In how many ways can 6 books be arranged on a shelf?

4. How many changes can be rung with 7 bells, if all are rung each time?

5. How many words each containing 4 different letters can be formed from the letters of the word *Yarmouth*?

6. If the number of permutations of n things 3 together is half of the number of permutations of $3n$ things 2 together, find n.

7. If ${}_nP_3 : {}_{n+2}P_5 = 1:30$, find n.

8. Find the number of arrangements of the letters of the word *algebra*, if the *a's* must be together.

9. In how many ways can 7 boys stand in a row? In how many if A and B must not be together?

10. How many signals can be made with 5 different flags, one above another, using (1) 3 at a time, (2) any number at a time?

11. In how many ways can 5 boys and 5 girls stand in a row, if no two boys are to be together?

12. How many numbers between 3000 and 5000 can be formed with the digits 1, 3, 5, 7, 9, no digit being repeated?

13. If the letters of the word *reason* be written in every possible way, how many will not begin with *re*?

273. To find the number of permutations of n things r at a time when each may be repeated any number of times.

Here there are n choices for the first thing, n for the second, n for the third and so on.

$$\therefore \text{ the required number} = n.n.n. \ldots \text{ to } r \text{ factors,}$$
$$= n^r.$$

Ex.—How many numbers of five digits can be formed from the digits 1, 2, 3, 4?

There are four choices for the first digit, 4 for the second and so on.
$\therefore$ the number $= 4.4.4.4.4 = 4^5 = 1024$.

274. When we are required to find the number of permutations of a number of things, it is understood that the things are all different. If some of them are alike the number of permutations is diminished.

Ex. 1.—Find the number of permutations of the letters *aaabcd*, taken all together.

If the 3 a's were different (like a_1, a_2, a_3) there would then be 6 different letters and the number of permutations would be $\underline{|6}$.

Any permutation as *baadac*, where the 3 a's are alike would make $\underline{|3}$ permutations if the a's were different, by merely permuting the a's, the other letters remaining fixed.

Therefore there are $\underline{|3}$ times as many permutations when the a's are different as when the a's are alike.

$$\therefore \text{ the required number} = \frac{\underline{|6}}{\underline{|3}} = 6.5.4 = 120.$$

Similarly, the number of permutations of the letters *aaabbcd* taken all together is $\dfrac{\underline{|7}}{\underline{|3}\ \underline{|2}}$ or 420.

By the same method of reasoning it follows that the number of permutations of n letters, taken all together, of which p are a's, q are b's and r are c's $= \dfrac{\underline{|n}}{\underline{|p}\ \underline{|q}\ \underline{|r}}$

Ex. 2.—Find the number of ways in which the letters of the word *Mississippi* may be written down.

Here there are 11 letters: 4 i's, 4 s's, 2 p's and 1 m.

$$\therefore \text{ the number} = \frac{\underline{|11}}{\underline{|4}\ \underline{|4}\ \underline{|2}} = 34{,}650.$$

EXERCISE 174

1.* How many numbers each of three digits can be formed from the nine digits, if the digits may be repeated?

2. Find the number of permutations of the letters, all together, of the words, *Halifax*, *Malacca*, *perimeter*, *indivisible*.

3. How many different numbers can be formed using all of the digits 1, 1, 1, 2, 2, 3, 4?

4. In how many ways can the letters of the word *Canada* be written, if each begins with *C* and ends with *a*?

5. How many even numbers each with 6 digits can be formed with the digits 2, 3, 3, 5, 5, 5?

6. How many permutations are there of the letters of the word *serene*, taken all together? How many of them begin with *e*? With 2 *e*'s? With 3 *e*'s?

7. Find the number of ways in which the 9 digits may be written, if the odd digits must be in ascending order. (Suppose the odd digits are all alike.)

8. In how many ways can 3 similar copies of an algebra, 4 of a geometry and 2 of an arithmetic be given to 9 boys, if each boy gets one?

9. In how many ways can 4 prizes be awarded to 4 boys if each boy may receive any number of the prizes?

278. Combinations. Each different group or selection which can be made from a number of things, in which the order of arrangement in the group is not considered, is called a **combination.**

Thus, the combinations of the letters *a*, *b*, *c*, *d* taken
one at a time are *a*, *b*, *c*, *d*;
two ,, ,, ,, ,, *ab*, *ac*, *ad*, *bc*, *bd*, *cd*;
three ,, ,, ,, ,, *abc*, *abd*, *acd*, *bcd*;
four ,, ,, ,, ,, *abcd*.

The number of combinations of n things r at a time is represented by the symbol ${}_nC_r$.

Thus, ${}_4C_1 = 4$, ${}_4C_2 = 6$, ${}_4C_3 = 4$, ${}_4C_4 = 1$.

276. To find the number of combinations of n different things taken r at a time.

Each combination of r things would make $\underline{|r}$ permutations by changing the order of the things in the combination and therefore there are $\underline{|r}$ times as many permutations as combinations.

$$\text{But } {}_nP_r = n(n-1)(n-2)\ldots\ldots(n-r+1),$$

$$\therefore \quad {}_n\mathbf{C}_\mathbf{r} = \frac{\mathbf{n(n-1)(n-2)\ldots\ldots(n-r+1)}}{\underline{|\mathbf{r}}}.$$

$$\text{Thus, } {}_{20}C_3 = \frac{20.19.18}{\underline{|3}} = 1140.$$

$$\text{And } {}_{20}C_{14} = \frac{20.19.18.\ldots\ldots8.7}{\underline{|14}}.$$

The last result may be put into a simpler form by multiplying each term by $\underline{|6}$ when we have

$${}_{20}C_{14} = \frac{\underline{|20}}{\underline{|14}\;\underline{|6}}.$$

Similarly in the general formula

$${}_nC_r =$$

$$\frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{\underline{|r}} \times \frac{(n-r)(n-r-1)\ldots\ldots3.2.1}{\underline{|n-r}},$$

$$\therefore \quad {}_n\mathbf{C}_\mathbf{r} = \frac{\underline{|\mathbf{n}}}{\underline{|\mathbf{r}}\;\underline{|\mathbf{n-r}}}.$$

277. The number of combinations of n things r at a time is the same as the number $n-r$ at a time.

$$\text{For } {}_nC_{n-r} = \frac{\underline{|n}}{\underline{|n-r}\;\underline{|n-(n-r)}} = \frac{\underline{|n}}{\underline{|n-r}\;\underline{|r}} = {}_nC_r.$$

$$\text{Thus } {}_{20}C_{18} = {}_{20}C_2 = \frac{20.19}{2} = 190.$$

Ex. **1.**—In how many ways can a team of 9 players be selected from 13 players?

As the order of selection is not considered,

$$\text{the number of ways} = {}_{13}C_9 = {}_{13}C_4 = \frac{13.12.11.10}{\underline{|4}} = 715.$$

If the factorial form is used, it is immaterial whether we use ${}_{13}C_9$ or ${}_{13}C_4$ for each is $\dfrac{\underline{|13}}{\underline{|9}\,\underline{|4}}$ and in simplifying the $\underline{|9}$ should be divided out.

Ex. **2.**—Find the number of lines which can be formed by joining 15 points in every possible way, if no three points are in the same straight line.

The number of lines is the same as the number of ways of selecting 2 points from 15 and is ${}_{15}C_2$ or 105.

Ex. **3.**—In how many ways can a party of 6 privates and 2 officers be chosen from 8 privates and 4 officers?

The 6 privates can be chosen from 8 in ${}_8C_6$ ways.

The 2 officers can be chosen from 4 in ${}_4C_2$ ways.

Therefore the number of ways the party can be made up

$$= {}_8C_6 \times {}_4C_2 = 28 \times 6 = 168.$$

Ex. **4.**—How many words consisting of 3 consonants and 2 vowels can be formed from 21 consonants and 5 vowels?

The 3 consonants can be selected in ${}_{21}C_3$ ways and the 2 vowels in ${}_5C_2$ ways, therefore the 3 consonants and 2 vowels can be selected in ${}_{21}C_3 \times {}_5C_2$ ways.

But the same 5 letters in any group may be arranged in $\underline{|5}$ ways, or the same 5 letters will make $\underline{|5}$ words.

$\therefore$ the number of words $= {}_{21}C_3 \times {}_5C_2 \times \underline{|5} = 1,596,000.$

EXERCISE 175

1.* Find the number of combinations of (*a*) 5 things 3 at a time, (*b*) 8 things 6 at a time, (*c*) 20 things 4 at a time.

2. Find the values of ${}_6C_3$, ${}_{15}C_{12}$, ${}_{13}C_5$, ${}_{10}C_7$.

3. How many different committees can be formed by selecting 3 persons from 12?

4. If the number of combinations of $2n$ things 3 at a time is 11 times the number of combinations of n things 3 at a time, find n.

5. Find the value of ${}_{100}C_{98} - {}_{99}C_{97}$.

6. If ${}_{n+1}P_3 : {}_nC_4 = 36 : 5$, find n.

7. If ${}_nC_5 = {}_nC_{12}$, find ${}_nC_{15}$ and ${}_{20}C_n$.

8. If 8 points, no three of which are in a straight line, be joined how many lines are formed?

9. How many diagonals are there in a polygon which has 8 sides? n sides?

10. How many triangles are there whose angular points are on n given points, no three of which are in a straight line?

11. From 8 men and 9 women in how many ways can a committee comprising 5 of each sex be formed?

12. A man has 20 friends, 10 of whom are relatives. In how many ways may he invite 15 guests from among them so that only 8 of them are relatives?

13. Out of 16 consonants and 5 vowels how many words can be formed containing 4 consonants and 1 vowel?

14. In how many ways can a watch of 5 be chosen from 3 officers and 12 men so as to include at least one officer?

15. Find the total number of combinations which can be formed from 5 things taking any number at a time.

16. Show that ${}_{n+1}C_r = {}_nC_r + {}_nC_{r-1}$.

17. In how many ways can a picket of 6 men be chosen from 15 men (1) if A must be chosen, (2) if A must not be chosen?

278. To find the value of r for which ${}_nC_r$ is greatest.

$$ {}_nC_r = \frac{n(n-1)(n-2)\ldots\ldots\ldots(n-r+2)(n-r+1)}{|\underline{r}}. $$

$$ {}_nC_{r-1} = \frac{n(n-1)(n-2)\ldots\ldots(n-r+2)}{|\underline{r-1}}. $$

$$ \therefore\quad {}_nC_r = {}_nC_{r-1} \times \frac{n-r+1}{r} = {}_nC_{r-1} \times \left(\frac{n+1}{r}-1\right). $$

Since $\frac{n+1}{r}-1$ decreases as r increases, the value of ${}_nC_r$ will continually increase until $\frac{n+1}{r}-1$ become 1 or less than 1, after which the value of ${}_nC_r$ will decrease.

$$ \therefore\quad {}_nC_r > {}_nC_{r-1} \text{ as long as } \frac{n+1}{r}-1 > 1, $$

$$ \text{or } \frac{n+1}{r} > 2, $$

$$ \text{or } \frac{n+1}{2} > r, $$

$$ \text{or } r < \frac{n+1}{2}. $$

(1) If n is even, the greatest value that r can have and be less than $\frac{n+1}{2}$ is $\frac{n}{2}$.

$\therefore$ ${}_n\mathbf{C}_\mathbf{r}$ is greatest when $\mathbf{r} = \frac{\mathbf{n}}{\mathbf{2}}$, if n is even.

(2) If n is odd, then $\frac{n+1}{2}$ is an integer, and when $r = \frac{n+1}{2}$, ${}_nC_r$ will equal ${}_nC_{r-1}$ and these two will be greater than any other.

$\therefore$ ${}_n\mathbf{C}_\mathbf{r}$ is greatest when $\mathbf{r} = \frac{\mathbf{n+1}}{\mathbf{2}}$ or $\frac{\mathbf{n-1}}{\mathbf{2}}$, if n is odd.

Thus, ${}_{20}C_r$ is greatest when $r = 10$.

${}_{21}C_r$ is greatest when $r = 10$ or 11.

EXERCISE 176 (Review of Chapter XXVIII)

1.* In how many ways can three prizes be distributed among 20 pupils, (1) if no pupil can receive two prizes, (2) if any pupil may receive all the prizes?

2. In how many ways may n boys stand in a row, if A and B must not stand together?

3. A boy has the choice of 5 routes to go to school and 3 to return. In how many ways may he go and return?

4. In how many ways can 8 persons be seated at a round table, (1) when the seats are distinguished, (2) when relative position only is considered?

5. How many numbers each of 4 digits can be formed from the nine digits (1) if no digit is repeated, (2) if the digits may be repeated?

6. How many different single tickets are required on a railroad having 25 stations?

7. Find the total number of permutations (with repetitions) of 6 things, not more than 4 being taken at a time.

8. How many more permutations are there of all the letters of the word *cannon* than of the word *canon*?

9. How many numbers greater than a million can be formed by using the digits 1, 2, 3, 4, 5, 6, 0, without repetitions?

10. There are n indefinite straight lines, no two of which are parallel and no three pass through the same point. How many intersections are there?

11. For what value of r is (1) ${}_{2n}C_r$, (2) ${}_{2n+1}C_r$ greatest?

12. If a polygon has 44 diagonals, how many sides has it?

13. In how many ways can a committee of 5 be chosen from 3 masters and 10 pupils, so as always to include at least one master?

14. In how many ways can 10 books be divided equally between two boys?

15. In how many ways can 9 books be divided among A, B and C, giving 2 to A, 3 to B and 4 to C?

16. In a group of 12 hockey players only one can keep goal. In how many ways may a team of 7 be chosen from the 12 so as always to include the goal keeper?

17. In how many ways can a pack of 52 cards be divided equally among 4 players?

18. A committee of 5 is to be chosen from 7 Conservatives and 4 Liberals. In how many ways may it be done so that there will be a Conservative majority on the committee?

19. A box contains 15 balls numbered from 1 to 15. In how many ways can 6 balls be selected if 1 is always included and 2 and 3 are always excluded?

20. Show that in ${}_{2n}C_n$, the number of combinations in which a particular thing occurs is $\frac{1}{2}$ of the whole number of combinations.

21. In a town are three different letter boxes. In how many ways can 5 letters be posted?

22. If 8 men and 5 women apply for 5 different positions, 3 of which must be filled by men and 2 by women, in how many ways may the positions be filled?

23. Find the sum of all the numbers which can be formed from the digits 1, 2, 3, 4, 5, using all the digits but without repeating any digit.

24. Show that the greatest number of combinations of $2n$ things, each containing the same number, is always double the greatest number that can be formed with $2n-1$ things.

ANSWERS

TO

HIGH SCHOOL ALGEBRA

ANSWERS

No answers are given to elementary examples, oral examples or examples which may be verified or checked without difficulty. In each exercise the number of the first example to which the answer is given is marked with a star.

Page 8

15. 108, 38, 10, 32, 60. **16.** 3, 14, 39, 0. **17.** 9, 29, 18. **19.** 2.
21. 25. **22.** 44, 7. **23.** 154, 616.

Page 10

9. 47. **10.** 70. **12.** $10x+10$. **13.** x ft. E. **14.** $15x$.
16. $2a^3+2a^2+3a$. **17.** 11, $5x$.

Page 12

24. 37. **25.** 17. **26.** 34. **27.** 1. **28.** 1. **29.** ½.

Page 14

30. $\left(\frac{a}{3}+\frac{b}{12}\right)$ hours. **31.** $(5x+20y-7z)$ cents. **32.** $\frac{a+2b}{3}$ cents.
33. $5x+10y+50z$. **34.** 1234, 4019. **35.** ·3, ·02, 2, 15, ·05, ·03.
36. $\frac{1}{x}+\frac{1}{y}$. **37.** 20, 20. **38.** 24. **39.** 2½.

Page 38

7. $7a+6b-15c$. **8.** $10x^2-14x+9$. **9.** $10a-7b$.
10. $4a+4b+4c$. **11.** $4a-8b+3c-5d$. **12.** $8x-6y+5z$.
13. $8a-4b$. **14.** $6a^2+8b^2-6c^2$. **15.** $3a+3b+3c+3d$.
16. $2x+2y-z$.

Page 40

18. 0. **19.** $15x+5y$. **20.** $4bc$. **21.** $10a^2+ab$. **22.** $4y^2$.
23. $2a-\frac{2}{3}b-\frac{1}{2}c$.

Page 41

13. $-3xy$. **14.** 0. **15.** $4p^2$. **16.** $10m-3n$. **17.** $6y-4z$.
18. 0. **19.** z.

Page 44

15. $3a-2b$. **16.** $2a+5c$. **17.** $-3x^3$. **19.** $4a^2-4a-16$.
20. $2b$. **21.** $13r-p$. **22.** $3b-5c-2a$. **23.** x^2+6x-5
24. $2a^2+a-12$. **25.** $a+b+c$. **26.** $2x-3$.
27. $10x^3+2x^2+8x+2$.

Page 46

11. $3x+2y$. **12.** $-2a-3b$. **13.** $a+b$. **14.** $b-a$.
15. $3a+b-3c$. **16.** $3x^2-3$. **17.** 7. **18.** 11.
22. 6, 4, 4, 6, 10. **23.** $4a+4b-15c$, $4a-4b+4c-4d$, y, 0.

Page 47

1. a **2.** $4x+6.$ **3.** $5n.$ **4.** $5c.$ **5.** $4b,\ 8a.$
6. $2a+2b+6c.$ **7.** $-x+y-5z.$ **8.** $3a-2b-2c.$ **9.** 14, 6.
10. $-7.$ **11.** $y-x.$ **12.** 31. **13.** $a-2c,\ 2c-a.$
14. $9m-2n.$ **15.** $4x-9.$ **16.** $x-12.$ **17.** $2b-4c.$
18. $5b-5a,\ a+3b-4c,\ 7a-b-6c.$ **19.** $\cdot 2x.$ **20.** $3x-6.$
21. $a-\frac{1}{6}b+\frac{1}{2}c.$ **22.** $5z-3x.$ **25.** 20. **26.** $1+2x.$ **27.** 7.
28. $7n+4x-2m.$ **29.** $3-a-b-c.$ **30.** $5c-3b.$

Page 52

19. 1, 4, 5, -3, -1, -8, -9, 7. **20.** 3, 16, 35.
21. $a^2,\ -a^2,\ -8,\ -1,\ 1,\ 81,\ 32.$ **22.** 29, 81. **23.** 24.
25. 90. **26.** 6. **27.** 30. **28.** 23. **29.** $-20.$ **30.** $-50.$
31. $-100.$

Page 53

13. $3a+7b+9c.$ **14.** $x-4y.$ **15.** $3m.$ **16.** $9a-b.$
17. $4a+\frac{1}{4}b.$ **18.** $6x^2+8x.$ **19.** $a^2.$ **20.** $x^3-9x^2+10x.$
21. $-4ab.$ **27.** $6x^2-15x.$ **28.** $7a^2-5a.$
29. $2xy,\ 3x^2+xy+3y^2,\ x^2+5xy+y^2.$

Page 56

19. $2x^2+4x-4.$ **20.** $5a^2-8a-22.$ **23.** 214. **24.** $4a^2-9b^2.$
25. $2x^2+2y^2,\ 4xy.$ **26.** $a^2+ab+4b^2.$ **27.** $14x+30.$
28. $x^2-6x-7.$ **29.** $2x-10.$ **31.** $12x^2+12.$ **32.** $3x^2+10.$
33. $15a.$ **34.** $3x^2+12x+14.$

Page 60

18. 1. **19.** x^2-3x+2. **20.** $2y$. **21.** 5. **22.** $a-b$.
23. a^2. **24.** $x+13$.

Page 60

2. 9, 16, -12, 25, -7, 27, -64, 91. **3.** 1, -1, 1, 16, -27.
4. $4a^2$. **5.** $8a^2-9a$. **6.** $3a^2-3b^2$. **8.** $30a+40b$.
9. $12x^2+12$. **10.** $4x^2+12xy-9y^2$. **13.** $13m^2+13n^2-24mn$.
16. $4x^2$. **18.** $5a^2-3ab-4b^2$. **20.** $3a^2-12a+14$.
21. $6x^2-2xy-6y^2$. **22.** $4-a$. **23.** 8, 19. **24.** x^4-16.
25. $8a^2-9a-1$, $6-10a$, $3a-4$. **26.** a^2-b^2, 2, a^2-b^2.
27. $20b^2-5bc$. **28.** 0.

Page 81

22. 7, -2. **23.** -8. **24.** 5, -2. **25.** 5, 6. **26.** $\frac{1}{2}$, 2.
27. 6, 1.

Page 83

21. 4, 5. **22.** -3, -3. **23.** 4, 9. **24.** 5, 3. **25.** 12, 12.
26. 19, 3. **27.** 15, -56.

Page 92

28. $2a$. **29.** 1. **30.** $x+1$. **31.** $3x-8$. **32.** $x+5$, $a+b$.
33. $2(x-2)(x-3)$. **34.** $3(a+4)(a-3)$. **35.** $x(x-7)(x-1)$.
36. ± 5, ± 1.

Page 93

17. 7, -11. **18.** 10, -4. **19.** 5. **20.** 10 in. **21.** 7 in.
22. 14 in. **23.** $3\frac{1}{2}$ in.

Page 95

17. $2x^2+2$. **18.** $2a^2+2b^2$. **19.** $5x^2+5$. **20.** $4ab$.
21. $5m^2-10mn$. **22.** $5x^2+24xy-5y^2$. **23.** $3x^2+12x+14$.
24. x^2-4. **25.** $18a-15$. **26.** $3x^2-4xy+6y^2$. **27.** $16x-34$.
28. $36x$. **29.** $9a^2-8ab+9b^2$. **31.** 8.

Page 98

25. $5a^2-5$. **26.** $3a^2-8b^2$. **27.** 0. **28.** $19q^2-4pq$.
29. x^4-a^4. **30.** 15. **31.** $3(x+y)(x-y)$. **32.** $5(x+2)(x-2)$.
33. $a(a+1)(a-1)$. **34.** $m(x-a)(x+a)$. **35.** $5(1+3p)(1-3p)$.
36. $(x+y)(x-y)(x^2+y^2)$. **37.** $\pi(R+r)(R-r)$.
38. $(x+1)(x-1)(a+b)$. **40.** $a^2-2ab-3b^2$ **41.** $2x$, 7.
42. 4, ±8.

Page 102

22. $x(x+1)(x-1)$, $3(x-2)(x+2)$, $a(a-1)(a-2)$. **23.** 8, −2 ; 2, −1.
24. 2. **26.** $2(x-2)(x+2)(x^2+4)$, $(a+2)(a-2)(a+3)(a-3)$,
$2m(m+3)(m-3)$, $(x+y)(x-y)(a+b)(a-b)$. **27.** $2b^2+2c^2+4$.
28. $3x^2-5y^2$. **30.** $12ab-38b^2$ **31.** 43, 23, 17, 13.
32. $3x+6y$.

Page 104

13. $a+2b$. **14.** $a-b$. **15.** $m-n$. **16.** $x+y$. **17.** $m+2$.
18. $a-2$. **19.** $x-3$. **20.** $y-1$. **21.** $a+b$. **22.** $x-5$.
23. $2(3a+2b)$. **24.** $a(a-1)$. **25.** 2, 3.

Page 106

22. $\frac{x+1}{x}$. **23.** $\frac{y}{y-1}$. **24.** $\frac{x-2}{x-3}$. **25.** $\frac{m+3}{m}$. **26.** $\frac{a-b}{a+4b}$.
27. $\frac{x-y}{x+2y}$. **28.** $a+1$. **29.** $\frac{a^2+1}{2}$. **30.** x^2+1.

Page 107

10. $\frac{x}{6}$. **11.** 4. **12.** $\frac{bx}{ay}$. **13.** $\frac{x+1}{x+2}$. **14.** $\frac{a-3}{a-5}$. **15.** 1

Page 108

10. $a^2(a+1)$. **11.** $3x(x+2)$. **12.** $ab(b+c)$. **13.** $2(x^2-1)$.
14. $x(x+y)^2$. **15.** $(x+1)(x-1)(x-2)$. **16.** $ab(a-b)$.
17. $(a+b)(a-b)^2$. **18.** $x(x-1)(x+1)$. **19.** $4x(x-1)(x+1)$.
20. $(y-1)(y+1)(y-2)$.

Page 109

22. $\frac{3x+2}{6}$. **23.** $\frac{5x-3y}{8}$. **24.** $\frac{5x+y}{12}$. **25.** $\frac{x^2+y^2}{x^2-y^2}$. **26.** 0

27. $\frac{2a}{x}$. **28.** $\frac{-x}{6(x+2)}$. **29.** $\frac{2b-3a}{ab(a-b)}$. **30.** $\frac{6a+11}{(a+1)(a+2)(a+3)}$.

31. $\frac{2}{(a-1)(a+2)}$.

Page 111

2. $x-2, 60(x-2)$. **3.** $x+y, xy(x+y)$. **4.** $x-2, (x-2)(x+4)(x-5)$.

5. $\frac{a+b}{a}, \frac{x}{x-y}, \frac{3a}{4b}, \frac{b}{c}$. **6.** 4. **7.** $\frac{1}{4}$. **8.** $\frac{x}{x-3}, \frac{x+2}{x+3}, \frac{2(x-3)}{3(x-2)}, \frac{a+b}{a-b}$

9. $\frac{2a}{(x+y)^2}$. **10.** $\frac{a-2x}{a^2}, \frac{x-1}{x-2}$. **11.** $\frac{7x}{12}$, 1. **12.** 0.

13. $\frac{b-4c}{4bc}$. **14.** $\frac{2}{15}, \frac{2x-z}{xz}$. **15.** 0. **16.** $\frac{1}{20}$. **17.** $\frac{4a}{a^2-b^2}$.

18. $\frac{x-2}{x-3}$. **19.** 1. **20.** $(a+b)^2$. **21.** 3. **22.** +4.

Page 116

18. $7x+6.$ **19.** $a+b.$ **20.** $3-x-y.$ **21.** $3x+3y+3z.$
22. $22x^2+24x-11.$ **23.** $10x^2-5x+15.$ **24.** $x^2+8x-12$
29. $a.$ **30.** 2 **31.** $4\frac{26}{29}.$

Page 119

17. $3x^2-12x+11.$ **18.** 0. **19.** $2ad+2bc.$ **20.** 0.
21. $12x^2+12.$ **22.** $1-x^6.$ **23.** $x^4-10x^3+35x^2-50x+24.$
24. $x^4-10x^2+9.$ **25.** $a^6-1.$ **26.** 13. **27.** 0.
28. 0. **29.** $1+3x+6x^2+10x^3.$ **30.** $2x^3+9x^2+3x-1.$
31. $28x^5+x^4y-33x^3y^2+31x^2y^3+20xy^4-12y^5.$ **33.** $2-x+4x^2-2x^3.$
34. 195. **35.** $5\frac{1}{2}.$ **36.** $abx^4+x^3(b^2-ac)+adx^2+x(bd-c^2)+dc.$
37. $p^2x^3+x(pr-q^2)+qr,\ x^3(a^2-a)+x^2(a^2+a-1)-1.$
38. $2a^2y^3-2b^2y+2bc$ **39.** $px^2+x(p^2+3p+3).$

Page 123

30. $x^2+2x+1.$ **31.** $-x^3+9x^2-1.$ **32.** $a^2+a+1,\ a^2-a+1.$
33. $-2xy.$ **34.** $2a.$ **35.** 6. **36.** $a-2.$ **37.** $a^2+3a-2.$
38. $x^2+xy+y^2.$ **39.** $2ax.$ **40.** $x+c.$ **41.** $x+p-1.$
42. $ax-b-c$ **43.** $ay+a+1.$

Page 125

1. 4. **2.** $-5.$ **3.** 22. **4.** $y+5.$ **5.** $2y^3.$ **6.** $-2y^3.$
7. $1+\frac{1}{x+1}.$ **8.** $1+\frac{3b}{a-b}.$ **9.** $2-\frac{5b}{a+b}.$ **10.** $5x-3+\frac{3}{x+2}.$
11. $1+x+x^2+x^3.$ **12.** $1-x+x^2-x^3.$ **13.** $1+2x+2x^2+2x^3.$
14. $1+2a+3a^2+a^3.$ **15.** $a-3.$ **16.** 6. **17.** $x-4,\ 7.$

Page 126

15. $a^3+b^3+c^3-3abc$. **16.** a^8-256. **17.** x^2-4x+8. **19.** 21.
20. $11x^2-7x-8$. **22.** $x^2(c-a)+x(d-b)+(f-c)$. **23.** 7.
26. $5y^4$. **27.** $x+6y-2z$. **28.** $2(ab+bc+ca)$.
29. $bx^3-bcx^2+x(ac-a+b)-bc$. **30.** 35. **31.** x^4-4x^2+12x.
32. $3x-8$. **33.** $3\frac{1}{9}$. **34.** 9. **35.** 6. **36.** p^2+p-2.
37. -52. **38.** 4. **40.** $x^4+2x^3+3x^2+2x+1$. **41.** 9.
45. $x^4+x^3(b+p)+x^2(q+bp+c)+x(bq+pc)+cq$. **46.** $3a+2b-c$.
47. $x^4+x^3+x^2(a-a^2)+x(1-2a)-1$. **48.** $a^3+2a^2bc+4ab^2c^2+8b^3c^3$.
49. $x^3-x^2(a+b+c)+x(ab+bc+ca)-abc$. **50.** $b-c$.
51. $3x^5-10x^4+3x^3-14x^2-7x$. **52.** $x^2+y^2+xy-2x-4y+4$.

Page 129

21. $3(x-2)(x^2+1)$, $a(x-1)(y-1)$. **22.** $a+b$, $x-1$.
23. $(2x-y)(5x-3z)$, $ab(a+c)(a-3b)$. **24.** $x-3$.
25. $(6x-7y)(8a+5b)$. **26.** $(x+y)(x+y+4)$, $(a-b)(2a-2b-1)$.

Page 131

34. $14x^2+19y^2$. **35.** $3a^2+3b^2+3c^2$. **36.** $2x^4+6x^2+2$.
37. $3a^2+3b^2+3c^2-2ab-2ac-2bc$. **38.** $8(x^2-z^2-xy+yz)$.
45. $3(x+1)^2$, $a(a+2b)^2$. **46.** $(a+b+2c)^2$, $(a+b-c-d)^2$. **48.** 3.
49. 14. **50.** $(x^2+y^2)(a^2+b^2+c^2)$. **51.** $(ax+by)^2+(ay-bx)^2$
52. 0.

Page 134

11. $a^2-b^2-c^2+2bc$. **12.** $4x^2+12xy+9y^2-25$.
13. $p^2-4q^2-9r^2+12qr$. **14.** $1+x^2+x^4$.
15. $a^2-b^2+c^2-d^2-2ac-2bd$. **16.** $a^2+4b^2-c^2-4d^2-4ab+4cd$.
44. $2(x+2)(x-2)$, $a(a+1)(a-1)$, $(a-x)(a+x)(a^2+x^2)$.
45. $5(a-b+2c)(a-b-2c)$, $(x-3b)(x-b)(x-5b)$.

Page 184 (*continued*)

46. $(b+c)(b-c)(a+d)(a-d)$, $(a+b+c)(a+b-c)(a-b+c)(a-b-c)$.
47. $(ax+c+by)(ax+c-by)$, $(m-n+3mn)(m-n-3mn)$.
48. $(x+1)(x-1)(3x-2)$, $x(x-1)(x-3)(x+3)$.
49. $2a^2-2ab+2bc-2c^2$. **50.** $(x+y)(x-y)(x+y+a)(x+y-a)$
51. $2a^2-6a+1$, $12xz-24yz$, $24a+9a^2-6a^3$, $20x^2y^2-40x^3y$.
52. $a^4+b^4+c^4-2a^2b^2-2b^2c^2-2c^2a^2$. **53.** $(x-y)(y-z)(z-x)$.
54. $(a-b)(c-a)$.

Page 137

13. $2(x^2+2x+2)(x^2-2x+2)$, $x(x^2+x+1)(x^2-x+1)$.
14. $(a-b)(a+b)(3a-b)(3a+b)$.
15. $(x^2-x+1)(x^2+x+1)(x^4-x^2+$ [illegible]).
16. $(a+b+c)(a+b-c)(a-b+c)(a-b-c)$. **17.** $(a^2+3)(3a^2+1)$.

Page 139

31. $3(x+8)(x-9)$. **32.** $2(a+1)(a+3)$. **33.** $x(3x-1)(2x-1)$.
34. $(x+1)(x-1)(x+2)(x-2)$. **35.** $a(a-1)(a+1)(a-3)(a+3)$.
36. $(a+1)(a-1)(3a+1)(3a-1)$. **37.** $(x+1)(x+3)(x-1)(x+5)$.
38. $(x-2)(x-7)(x+1)(x-10)$. **40.** x^2-5x+6. **41.** $4x^2-16x+15$.
42. ± 1, ± 11, ± 19, ± 41. **43.** $33a^2-38ab-8b^2$.
44. $(x+y)(x+4y+1)$. **45.** $(3a+2b)(a-b+2)$. **46.** x^2+1.

Page 141

22. $2(a-2)(a^2+2a+4)$. **23.** $3(y+3)(y^2-3y+9)$.
24. $a(a+1)(a^2-a+1)$. **25.** $b(a+b)(a^2-ab+b^2)$.
26. $(a^2+b^2)(a^4-a^2b^2+b^4)$. **27.** $(x+y+a)(x^2+2xy+y^2-ax-ay+a^2)$.
28. $x(x^2-6x+12)$. **29.** $(2a-b)(a^2-ab+b^2)$. **30.** $2a(a^2+3b^2)$.
31. $x+y$. **33.** $a^2(a^4-6a^2bc+12b^2c^2)$, $y^2z(3x-yz)(9x^2+3xyz+y^2z^2)$.
34. $(a-b)(a+b)(a^2+b^2)(a^2-ab+b^2)(a^2+ab+b^2)(a^4-a^2b^2+b^4)$.
35. $(x+1)(x-2)$. **36.** 2.

Page 143

7. $(x-1)(2x^2-9x-4)$. **8.** $(x-1)(x+1)(x-2)$.

9. $(x-1)(x-2)(x+3)$. **10.** $(x-2)(x-3)(x+5)$.

11. $(a-1)(a-2)(a+4)$. **12.** $(a+b)^2(a-2b)$. **16.** -12.

17. $(a-b)(a+2b)(a+3b)$. **20.** 8, -4. **21.** 2, 3.

Page 147

4. $2x^2+2a^2-2b^2$. **5.** $4x^2$. **6.** $4a^2+4b^2+4c^2$.

7. $14a^2+14b^2+14c^2+14ab-10ac-22bc$. **8.** 19997.

9. 14,860,000. **10.** -5. **11.** $2a+197$. **30.** $(x-2)(4x-9)$.

31. $3a(2a-b)(4a^2+2ab+b^2)$. **32.** $8(a+c)(c-a-b)$.

33. $(3x-4)(4x+5)$. **34.** $4(3a-5)(9a^2+15a+25)$.

35. $(x+y)(x-y+1)$. **36.** $(x-3)(x+3)(x^2+2)$.

37. $(x-y)(x^2-xy+y^2)$. **38.** $(x+11y)(x-12y)$.

39. $(a-b+c)(a-b-c)$. **40.** $(x+y)^2$. **41.** $(x-3b)(ax-2)$.

42. $(a+2b)(a-2b-3)$. **43.** $(2x-y)(2x+y+a)$.

44. $(a+b)(a+b+c)$. **45.** $(a-b)(a-b-1)$.

46. $(x-y)(x^2+xy+y^2+x+y+1)$. **47.** $ab(a+b)(a-b)^2$.

48. $(2a+5b)(2a-5b+1)$. **49.** $9b(4a^2+2ab+b^2)$.

50. $(x^2+4xy-y^2)(x^2-4xy-y^2)$. **51.** $(a^2-b^2+a-3)(a^2-b^2-a+3)$.

52. $(x-1)(x^2-10x-3)$. **53.** $(a-1)(3a^2-2a-10)$.

54. $(x+1)(x-1)(c+1)(c^2-c+1)$.

55. $(a+1)(a-1)(a+2)(a^2+1)(a^2-2a+4)$. **60.** $(a-b)(b-c)(c-a)$.

61. $(x-2)(2x+3)(3x-2)$.

62. $(x-1)(x-2)(x-3)(x-4)$, $(x+1)(x+3)(x-2)(x-6)$.

63. 1, 5, -6; 0, 1, 6, -7. **65.** x^2-c^2. **66.** $a^2-b^2+c^2+2ac$.

67. -4, 5. **68.** $(x-a)(2x+a+b)$.

69. $(ab+cd)^2-(ac+bd)^2$, $(ab-cd)^2-(ac-bd)^2$.

Page 153

16. 30. **17.** 13. **18.** 1, 3, 10. **19.** 29. **20.** 1, 3, 5.
21. 16.

Page 155

6. 45. **7.** 4. **8.** 1.

Page 169

6. About 2 h. 35 m. after *A* started; 31 m. from Toronto.
7. (*a*) At 10.55, 2 m. from *C* towards *D*. (*b*) 22 m., 17 m. (*c*) 11.10.

Page 172

5. 13. **6.** A square, 16. **7.** Right-angled, 4. **8.** *M*.
9. 54. **10.** 16; 5, 6½, 8. **11.** 6.

Page 177

3. 5, 10, 13. **4.** 6¾. **5.** 13. **6.** 7¾, 4. **7.** 30, 30.
8. (1, −7), (−3, −17), (5, 3). **9.** 112½. **10.** 24. **17.** 24.
18. (4, 4). **25.** (3, 2), 90°. **26.** $1200, 12th.

Page 181

1. $4xy^2z$, $24ax^3y^4z^3$. **2.** $x-y$, $xy(x^2-y^2)$. **3.** $a+b$, $b(a-b)(a+b)^2$.
4. $x-3$, $(x-3)(x-4)(x+3)(x+5)$. **5.** $a+5$, $(a+5)(a+3)(a-7)(a-2)$.
6. $3(x-2)$, $3(x+1)(x+2)(x-2)^2$. **7.** $x-y$, $y(x-y)(x+z)$.
8. $m-2$, $4m^2n^2(m+2)(m-2)^2(m^2+2m+4)$.
9. $2(a^2+ab+b^2)$, $6a(a^3-b^3)$. **10.** $a+b-c$, $a(a+b-c)(a+b+c)$.
11. $a+b+c$, $(a+b+c)(a-b-c)(b-c-a)(c-a-b)$.

Page 181 (*continued*)

12. x^2-xy+y^2, $(x+y)(x^4+x^2y^2+y^4)$.
13. $3x-2$, $(3x-2)(x+3)(x-3)(2x-3)$.
14. $5x-1$, $(5x-1)^2(5x+1)(2a+3c)$. **15.** $x-3$, $x(x-3)(x-2)(x^2+5)$.
16. $u-v$, $(u-v)(u+v)(u^2+v^2)(u^2+uv+v^2)$.
17. x^2-8, $(x^2-8)(x+2)(x+3)$. **19.** $-a$.
20. $x^2-3xy+2y^2$, $x^2+xy-6y^2$.

Page 182

1. $x-1$, $(x-1)(x-2)(x^2-5x+3)$. **2.** $a-1$, $(a-1)(a-5)(a^2-18a-1)$.
3. $x-2$, $(x-2)(x^2+4)(2x^2-3x-6)$.
4. $a-1$, $(a-1)(a^2+1)(3a^2+a+6)$.
5. $x-1$, $x(x-1)(x+4)(x^2+x-6)$. **6.** $(x-2)(x^2+5x+1)(x^2-2x-1)$.
7. $\dfrac{a-b}{a^2+2ab-15b^2}$, $\dfrac{1}{2x+4}$. **8.** $x^3-6x^2+11x-6$, $x^3-9x^2+26x-24$.

Page 186

1. $x-5$. **2.** $(a-3)(a-4)$. **3.** $2(3x^2+2x+2)$. **4.** $2x-9$.
5. $2b^2-b-5$. **6.** $3x-7y$. **7.** $a-2$. **8.** $x-3$
9. $3a^2(a-1)$. **10.** $x-1$ **11.** $(x-3)(x+1)(x+2)(x^2-x+1)$.
12. $(x+1)(x+2)(x+3)(x+4)$. **13.** $(2x+3)(3x-4)(x^2+3x-1)$.
14. $(x-1)(x-2)(x-3)(x-4)$. **15.** $(5x^2-1)^2(4x^2+1)(5x^2+x+1)$.
16. 3. **17.** 35. **18.** $x^2+5x-14$. **19.** 11.

Page 187

1. $x-11$, $(x-9)(x-10)(x-11)(x-13)$.
2. $x-3$, $(x-3)(x-12)(x^2-2)(x^2+3x+9$
3. $a-b$, $(a-b)^2(a+b)(a^2+ab+b^2)$.
4. $x+3$, $x(x+3)(x+2)(x-4)(x-5)$.
5. $(2a+1)(a-3)$, $(2a+1)(a-3)(a+3)(2a-1)$.

Page 187 (*continued*)

6. $x-b$, $(x-a)(x-b)(x-c)$. **7.** $x-1$, $(x-1)(x-2)(x-3)(x+2)(x+3)$.
8. $(x-1)(x+3)$, $(x-1)(x+3)(x^2+x+4)(x^2-6x-4)$.
9. $(a+3)(2a+1)$, $(a-2)(2a+1)(a+2)^2(a+3)^2$.
10. $(x-y)^2$, $(x-y)^2(x-2y)^2(x+2y)^2$.
11. x^2-xy+y^2, $(x^2-xy+y^2)^2(x^2+xy+y^2)$. **13.** 3.
14. $x^4-x^2a^2+a^4$. **17.** 1, 3. **18.** x^2-3x+2, x^2-6x+5.
19. $\frac{3x+2}{2x+3}$.

Page 191

7. $\frac{a+1}{a^2+a+1}$. **8.** $\frac{x}{x-2}$. **9.** $\frac{a-1}{4a^2+3a-6}$. **10.** $\frac{x+2}{2x^3(x+1)}$.
11. $\frac{x-1}{x+1}$. **12.** $\frac{x^2-2}{12x^2-7x-4}$. **13.** $\frac{a^2-3}{a^4-2a^3+2a-5}$. **14.** $\frac{x-3}{2x-1}$.

Page 192

1. $\frac{a^2+b^2}{a^2-b^2}$. **2.** $\frac{2y}{x^2-y^2}$. **3.** $\frac{4xy}{x^2-y^2}$. **4.** $\frac{6}{a^2-7a+10}$.
5. $\frac{2ab}{a^2-b^2}$. **6.** $\frac{2a^2}{1-a^4}$. **7.** $\frac{2xy}{x^2-y^2}$. **8.** $\frac{3x}{(x+y)(2x-y)}$. **9.** 0.
10. $\frac{3}{(x+4)(x+5)(x+7)}$. **11.** $\frac{3x^2-5xy-2y^2}{x^2-y^2}$. **12.** $\frac{5}{x^2-5x+6}$.
13. $\frac{y}{x-y}$. **14.** $\frac{2b^3}{a^2-b^2}$. **15.** $\frac{2x+2y}{x-y}$. **16.** 0. **17.** $\frac{1}{x^3-1}$.
18. $\frac{3}{2a-3b}$. **19.** $\frac{2}{(x-1)(x-2)(x-3)}$. **20.** 0. **21.** 2.
22. 0. **23.** 1. **24.** $\frac{a}{2(a+1)}$. **25.** $\frac{2y}{x^2-y^2}$, $\frac{4y^3}{x^4-y^4}$.
26. $\frac{8y^7}{x^8-y^8}$. **27.** $\frac{4x^3}{81-x^4}$. **28.** $\frac{4a^2}{a^2-b^2}$. **29.** $\frac{2x^5}{1-x^8}$. **30.** 2.

Page 195

1. $\frac{x-a}{ax(a+x)}$. **2.** $\frac{v}{2a-3b}$. **3.** $\frac{2x}{x-2}$. **4.** $\frac{-3y}{x^2-9y^2}$

5. $\frac{2b+3a}{ab(a-b)}$. **6.** $\frac{x^2+ax}{a(x-a)}$. **7.** $\frac{x^2-x+2}{x^2-1}$. **8.** $\frac{1}{x-2}$.

9. $\frac{x^2}{x^2-y^2}$. **10.** 0. **11.** $\frac{2}{b-3a}$. **12.** $\frac{8x}{3a+2x}$. **13.** $\frac{20}{x^2-1}$.

14. $\frac{a^2+c^2}{ac(a-b)}$. **15.** $\frac{1}{(c-a)(c-b)}$. **16.** $\frac{x}{(x-a)(x-b)}$.

17. $\frac{x+y}{x(x-y)^2}$. **18.** $\frac{1}{1-9x^2}$. **19.** 0. **20.** 0. **21.** c.

22. 1. **23.** $\frac{x^2+y^2+z^2-xy-yz-zx}{(x-y)(y-z)(z-x)}$. **24.** -1. **25.** b

26. d. **27.** $\frac{b^2}{(a+b)(a^2+b^2)}$. **28.** $\frac{32x}{(x^2-9)(x^2-25)}$

29. $\frac{-6}{(a+1)(a+2)(a+3)(a+4)}$. **30.** $\frac{16x}{(x^2-1)(x^2-9)}$.

Page 198

1. 1. **2.** $\frac{1}{a^2-b^2}$. **3.** $\frac{x-4}{x+4}$. **4.** $\frac{a^2+b^2}{a}$. **5.** $\frac{4ab}{a^2-b^2}$. **6.** $\frac{y^4}{x^2+y^2}$.

7. $x^4+1+\frac{1}{x^4}$. **8.** $a^4+\frac{4}{a^4}$. **9.** 1. **10.** 1. **11.** $\frac{x-6}{x-3}$.

12. $\frac{y(a-b)}{ax}$. **13.** $\frac{a+b-c}{a-b+c}$. **14.** $\frac{x}{y}+\frac{y}{x}+1$. **15.** $\frac{x^2}{y^2}-1+\frac{y^2}{x^2}$.

Page 198 (*continued*)

16. $\frac{x^2-xy+y^2}{x^2+y^2}$. **17.** $\frac{(a+1)^2}{(a-2)^2}$. **18.** $\frac{a^2-7a+10}{a^2}$. **19.** $\frac{x+1}{x-2}$.

20. $\frac{1}{a^2-x^2}$. **21.** $\frac{bc}{a^2}$. **22.** 1. **23.** $\frac{1}{a-8}$. **24.** $\frac{3(a+2b)}{a-6b}$.

Page 200

1. $\frac{a}{10bc}$. **2.** $\frac{72ac}{5b}$. **3.** $\frac{1}{a^2-b^2}$. **4.** $\frac{-1}{xy}$. **5.** $\frac{1}{x}$. **6.** $-\frac{a}{b}$.

7. $\frac{2xy}{x^2+y^2}$. **8.** $\frac{6}{a^2}$. **9.** 1. **10.** $\frac{xy+1}{xy(xy-1)}$. **11.** $\frac{b^2}{a^2}$.

12. $\frac{c^2-a^2}{a^2-b^2}$. **13.** $\frac{2}{x}$. **14.** $a+b$. **15.** $a+b$. **16.** $\frac{2ab}{a^2+b^2}$. **17.** $\frac{y^2}{x^2}$.

Page 201

1. $\frac{4a^4}{a^4-b^4}$. **2.** $\frac{ac^2-a}{bc}$. **3.** $\frac{2y^2}{x^2-9y^2}$. **4.** 0. **5.** $\frac{8}{1-x^2}$. **6.** $\frac{2}{x}$.

7. $\frac{1}{2}$. **8.** $\frac{1+3x^2}{2x}$. **9.** $\frac{b-c}{b+c}$. **10.** $\frac{x+2}{x^2+4x+3}$. **11.** $\frac{a+b}{a^2-ab+b^2}$.

12. $\frac{a^2+b^2}{a+b}$. **13.** $\frac{2(a+x)}{a^2+ax+x^2}$. **14.** $\frac{1+a}{2}$. **15.** 0. **16.** 1.

17. $\frac{3x-15}{(2x+3)(3x-2)}$. **18.** $\frac{1}{x-1}$. **20.** x. **21.** 1. **22.** 1.

25. $x+x^2$. **26.** $\frac{a}{b}$. **27.** -1. **32.** $\frac{a+2b}{a+b}$. **33.** $\frac{b^2}{(a+b)(a^2+b^2)}$

34. 0. **35.** $\frac{4x^2-10x}{(x-1)(x-2)(x-3)(x-4)}$. **36.** 0. **37.** $\frac{xy}{x+y}$.

Page 210

13. $\frac{cd-ab}{a+b-c-d}$. **4.** $\frac{-a}{a+b}$. **15.** $a-b$. **16.** $\frac{ab}{a+b-c}$.

17. $\frac{2ab}{a+b}$. **18.** $\frac{a^2+ab+b^2}{a+b}$. **19.** $\frac{5a}{2}$. **20.** $-\frac{a+b}{2}$. **21.** $\frac{ac}{b}$.

22. $\frac{bc^2}{a^2}$. **23.** $a-b$. **24.** $\frac{3b-a}{2}$. **25.** $\frac{ab}{b+c}, \frac{ac}{b+c}$.

26. $\frac{an+b}{m+n}, \frac{am-b}{m+n}$. **27.** $\frac{ab}{a-b}$. **28.** $\frac{3ab-3a^2}{a+3}$. **29.** $\frac{mn(a+b)}{mn-m-n}$.

30. $\frac{ab-cd}{a+b-c-d}$. **31.** $\frac{2s}{a+l}, \frac{2s-ln}{n}, \frac{2s-an}{n}$.

32. $s-sr+rl, \frac{sr-s+a}{\;}, \frac{s-a}{s-l}$. **33.** $\frac{2s-gt^2}{2t}, \frac{2s-2at}{t^2}$.

Page 213

13. $a, -b$. **14.** $\frac{c_1-b_1}{a_1-b_1}, \frac{a_1-c_1}{a_1-b_1}$. **15.** $2a, -b$. **16.** a, b.

17. $\frac{a_1b_2-a_2b_1}{b_2c_1-b_1c_2}, \frac{a_1b_2-a_2b_1}{a_1c_2-a_2c_1}$. **18.** $b+a, b-a$ **19.** $\frac{1}{2}a, \frac{1}{3}b$.

22. $c, 0, a$. **23.** $4a-3b$.

Page 214

25. $\frac{7ab-3a^2}{a-3b}$. **26.** abc. **27.** 5, 5. **28.** $21\frac{1}{11}, 27\frac{8}{11}$.

29. $2, -\frac{1}{3}$. **30.** $\frac{1}{5}, 2$. **31.** -3. **32.** \$543, \$457.

33. $\frac{41}{56}$. **36.** \$16400, \$13600. **37.** 1540, 880, 616.

38. $\frac{2s-n^2d+nd}{2n}, \frac{2s-2an}{n^2-n}$. **39.** $2a^2+2b^2$. **40.** 35.

41. \$2100, \$560. **42.** 182040.

Page 219

18. a^2+3a+1. **19.** 6. **20.** -4. **21.** $(x+1)(x+2)(x+3)$.
22. $1-x-2x^2$, $2-3x-\frac{9}{4}x^2$.

Page 221

19. $(x+y)^2-2(x+y)+1$. **20.** $x^2-3ax+a^2$. **21.** a^2+b^2.
22. a^2+b^2 **23.** $x^2+2+\frac{1}{x^2}$. **24.** -6. **25.** 13.

Page 223

21. $2a^3+6ab^2$, $6a^2b+2b^3$. **23.** $2a^3+2b^3+6a^2b+6ab^2+6ac^2+6bc^2$.
27. 27. **28.** 242.

Page 225

17. x^2+x+1. **18.** $1-2x+3x^2$. **19.** $\frac{x}{3}-1+\frac{3}{x}$.
20. $3a^2-4a+1$. **21.** $1-x^2$. **22.** $4c$ **23.** $x-1$.
24. $a-3$. **25.** $x-2$.

Page 226

1. $3x^2-4xy+2y^2$. **2.** x^3+2x^2-3x+1. **3.** $x^6+3x^4-2x^2+2$.
4. $\frac{1}{2}x^2-\frac{1}{3}x+1$. **5.** $5x^2-2ax-3a^2$. **6.** $2a^2+3a+7$.
7. $(x+2)(x+3)(x+4)$. **8.** $(x+1)(x-5)(2x-3)$. **9.** $2x^2-5x+2-\frac{3}{x}$.
10. $3-5x$. **11.** $2x^2-x+1$. **12.** a.
13. $1-x-\frac{1}{2}x^2$, $1-\frac{1}{2}a-\frac{1}{8}a^2$, $2+\frac{1}{4}x-\frac{1}{64}x^2$. **15.** $7x^2-2x-\frac{3}{2}$.
18. $8a^3$. **21.** 0, $-8y^3$. **24.** $2x^3-3x^2+x-2$. **25.** 16.
27. $6x-4$. **28.** $7x^2-2x+1$.

Page 230

32. $\sqrt{13}$. **34.** $\sqrt{21}$. **35.** $2\sqrt{2}$ $4\sqrt{2}$. **36.** $5\sqrt{2}$
37. $4\sqrt{2}$, $12\sqrt{2}$.

Page 232

9. $10\sqrt{3}$. **10.** $7\sqrt{2}$. **11.** $5\sqrt{5}$. **12.** $-3\sqrt{7}$. **13.** $7\sqrt{2}$.
14. $8\sqrt{11}$. **15.** $7\sqrt{5}$. **16.** $-4\sqrt{2}$. **17.** 8·66. **18.** 7·94.
19. 11·62. **20.** 5·20. **21.** −1·41. **22.** 25·46.
23. ±6·083. **24.** ±3·873. **25.** ±6·782. **26.** ±6·481.
27. ±9·592. **28.** ±13·711. **29.** 7·483.

Page 234

13. $24\sqrt{3}$. **14.** $12\sqrt{7}$. **15.** $5+2\sqrt{6}$. **16.** $27-4\sqrt{35}$.
17. $30+12\sqrt{6}$. **18.** $a+b+2\sqrt{ab}$. **19.** $2+3\sqrt{2}$.
20. $12+\sqrt{6}$. **21.** $6+\sqrt{10}$. **22.** $6a+6b-13\sqrt{ab}$.
23. $6+2\sqrt{15}$. **24.** $4\sqrt{6}-4$. **25.** $a+b-6-\sqrt{a+b}$. **26.** 1.
27. $6+2\sqrt{3}+2\sqrt{2}+2\sqrt{6}$. **28.** $16+4\sqrt{10}-2\sqrt{15}-4\sqrt{6}$.
29. $2a+2\sqrt{a^2-b^2}$. **30.** $13x-5y-12\sqrt{x^2-y^2}$. **31.** 1
32. $12-4\sqrt{2}$. **33.** $6\sqrt{6}$. **34.** 70. **35.** $30-5\sqrt{6}$.
36. $\sqrt{8}+\sqrt{7}$. **37.** 42, 43. **38.** 46. **39.** $9\sqrt{2}$.
40. 30·92.

Page 236

13. $14+8\sqrt{3}$. **14.** $6\sqrt{2}+4\sqrt{3}$. **15.** $5+2\sqrt{6}$. **16.** $\frac{a-\sqrt{ab}}{a-b}$.
17. $\sqrt{15}$. **18.** $\frac{\sqrt{7}-\sqrt{2}}{5}$. **19.** ·577. **20.** 3·536. **21.** ·81[illegible].

Page 236 (*continued*)

22. ·318. **23.** 1·491. **24.** ·084. **25.** 1·225. **26.** ·894.
27. ·072. **28.** 2·12. **29.** ·82. **30.** 1·39. **31.** 3·15.
32. 11·71. **33.** ±2·73. **34.** 1·008. **35.** $\sqrt{2}$.

Page 238

1. $12\sqrt{2}$. **2.** $12\sqrt{5}$. **3.** $10\sqrt{3}$. **4.** 62. **5.** 191.
6. $-4\sqrt{2}$. **7.** $22-12\sqrt{2}$. **8.** $12-4\sqrt{6}-2\sqrt{3}+4\sqrt{2}$. **9.** $\frac{1}{3}$.
10. $9-4\sqrt{5}$. **11.** 1. **12.** $2\sqrt{13}+2\sqrt{2}$. **13.** $74+11\sqrt{6}$.
14. $\frac{5}{6}$. **15.** 1·732. **16.** $\sqrt{12}+\sqrt{10}$. **17.** 1, 2.
18. $2\sqrt{2}$, $\frac{1}{2}\sqrt{6}$, $\frac{1}{4}\sqrt{30}$, $\frac{1}{2}\sqrt{14}$, $\frac{1}{5}(4\sqrt{2}-2\sqrt{3})$.
25. ±8·661, ±7·937, ±9·899, 1·291, ·518.
26. ·817, ·447, ·414, ·757, ·337. **27.** $25\sqrt{3}$. **28.** $2\sqrt{2}$
29. $\frac{13-\sqrt{5}}{2}$, $4\sqrt{6}$. **30.** 2·02. **31.** 30. **32.** 5. **33.** 4·83.

Page 241

1. $x^2+x-132=0$. **2.** $x^2-x-156=0$. **3.** $x^2-49=0$.
4. $x^2+6x-112=0$. **5.** $5x^2-6x-440=0$. **6.** x^2+6x-[illegible]$400=0$.
7. $x^2-19x+88=0$.

Page 242

1. 6, −1, −22. **2.** 6, −25, 21. **3.** 8, 19, −15. **4.** 12, −11, 2.
5. 1, −10, 9. **6.** 2, −5, 2. **7.** 1, 4, −32. **8.** 5, −27, 28.
9. 2, −19, 44. **10.** 2, −5, −3. **11.** 0, 2, 7. **12.** 0, 1, −1.

Page 248

1. 4·236, −·236. **2.** 7·828, 2·172. **3.** 1·646, −3·646.
4. 1·916, −9·916. **5.** ·232, −3·232. **6.** ·851, −2·351.
7. $3\pm\sqrt{11}$. **8.** $-4\pm3\sqrt{3}$. **9.** $\frac{1}{2}\pm\sqrt{2}$. **10.** $1\pm\frac{1}{2}\sqrt{41}$.
11. $\frac{5\pm\sqrt{157}}{6}$. **12.** $-\frac{1}{3}\pm\frac{1}{6}\sqrt{22}$.

Page 251

20. 1·618. **21.** $2\frac{1}{2}$, −1. **22.** 5, 12. **23.** 6, −2.
24. 14, 6 or 16, 4. **25.** $x=1$ or 2, $y=2$ or 1. **26.** 540.
27. 3·236, −1·236. **28.** 20c. **29.** 60, 90. **30.** 1·449, −·949.
31. 20. **32.** 3 m. per hr. **33.** 8. **34.** $\frac{a^2+ab}{a-b}$, $\frac{a^2-ab}{a+b}$.
35. 20, 30. **36.** $x=2$ or $\frac{1}{2}$. **37.** 4.

Page 255

18. 2·54, 1·0936. **19.** 8 : 27. **20.** ·192, 1·302.
21. 3937 : 6336. **22.** 4 : 5, 11 : 27, $a+3 : a+5$.
23. 11 : 15, 13 : 18, 2 : 3, 3 : 5. **31.** $\frac{ad-bc}{c-d}$. **32.** $\frac{1+3a}{1+4a}$.
33. 4 : 5. **34.** $\frac{ab}{b+c}$ $\frac{ac}{b+c}$. **35.** 10. **39.** $20bm : mn$.

Page 259

29. 11·55. **30.** $10\frac{1}{2}$, $4\frac{1}{2}$. **31.** $AE=6\frac{3}{4}$, $DE=7\frac{1}{2}$. **32.** 240.
33. 9·899. **34.** 2 : 3. **35.** 2, $\frac{d-b}{a-c}$, $\frac{m+n}{m-n}$, $-\frac{q}{p}$

Page 259 (*continued*)

36. $\frac{3}{2}$ or $\frac{2}{3}$, 5 or -1. **37.** $\frac{a}{5}=\frac{b}{3}=\frac{c}{-8}$.

38. $AC=20$, $AE=5$, $DE=4$. **39.** 147 ft. **40.** 3 or $\frac{1}{3}$.

43. $5:4:2$. **44.** $\frac{5}{3}$ or $-\frac{1}{3}$. **45.** $6\frac{2}{3}$, $5\frac{1}{3}$; $\frac{ac}{b+c}$, $\frac{ab}{b+c}$.

46. $2:3$. **47.** $17\frac{1}{2}$, 25, 30. **48.** $110:15:17$.

Page 262

6. 6; $7\frac{4}{5}$; $11\frac{1}{13}$, 5. $4\frac{8}{13}$ · 5. $2\frac{2}{5}$, $1\frac{4}{5}$. **8.** 2, 8. **9.** 3, 6, 12.

Page 264

4. $\frac{27}{44}$. **5.** $-17\frac{1}{3}$. **6.** $\frac{87}{97}$. **14.** $\frac{20a}{9}$. **15.** $\frac{11}{5}$.

17. $\frac{-c}{a+b}$. **18.** $7:16$.

Page 266

17. $\frac{ab}{a+b}$. **18.** $\frac{78}{5}$. **19.** 2, $\frac{4}{3}$. **20.** $\frac{5}{4}$. **21.** $\frac{3}{2}$, $\frac{5}{3}$. **22.** 5.

23. $\frac{x}{3}=\frac{y}{4}=\frac{z}{5}$. **24.** $\frac{2}{13}$. **25.** $xy-\frac{1}{xy}$. **26.** $\frac{1}{3}$.

32. ± 10, ± 5, ∓ 5. **38.** $41\frac{2}{3}$. **39.** 1, 3, 4.

Page 270

1. $3a$, $-a$. **2.** b, $-5b$. **3.** $3m\pm m\sqrt{6}$. **4.** $-2p\pm p\sqrt{5}$.

5. $a\pm\sqrt{a^2-b}$. **6.** $-b\pm\sqrt{b^2+c}$. **7.** $-1\pm\sqrt{\frac{b}{a}+}$

8. $\frac{-b\pm\sqrt{b^2-ac}}{a}$. **9.** $\frac{b\pm\sqrt{b^2+4ac}}{2a}$. **10.** $\frac{q\pm\sqrt{q^2-4pr}}{2p}$.

Page 271

1. 1, $\frac{3}{2}$. **2.** $\frac{3}{4}$, $\frac{7}{6}$. **3.** $\frac{1}{23}$, $-\frac{1}{23}$. **4.** $\frac{5}{2}\pm\frac{1}{2}\sqrt{11}$.

5. $\frac{4}{15}$, $-\frac{3}{13}$. **6.** $\frac{13}{4}\pm\frac{1}{4}\sqrt{89}$. **7.** $\frac{5}{17}$, $-\frac{7}{25}$. **8.** $\frac{1}{30}$, $-\frac{1}{40}$.

9. $2a$, $-3b$. **10.** 7, $\frac{11}{3}$. **11.** $\frac{1}{2}$, $-\frac{1}{3}$. **12.** $\frac{1}{40}$, $-\frac{1}{45}$.

19. $\frac{9}{10}\pm\frac{1}{10}\sqrt{161}$. **20.** $\frac{3}{2}\pm\frac{1}{6}\sqrt{57}$. **21.** $\frac{3}{2}$, $-\frac{1}{2}$. **22.** $\frac{1}{2}\pm2\sqrt{5}$.

32. $1\pm\frac{a}{2}\sqrt{2}$. **33.** $\pm\sqrt{6}$. **35.** $1\pm2\sqrt{6}$. **38.** 2·414, −·414.

39. 3. **43.** 6·18, 3·82. **46.** 2·786 or −·120

53. 7·03, 8·78, 8, 2·29, 6·42, impossible.

Page 276

13. 1, −6, $-\frac{5}{2}\pm\frac{1}{2}\sqrt{-39}$. **14.** 6, −3, $\frac{3}{2}\pm\frac{1}{2}\sqrt{-71}$.

15. 2, $-1\pm\sqrt{-3}$. **16.** ±2, $\pm2\sqrt{-1}$. **17.** 3, 2, −5.

18. 1, $\frac{2}{3}$, $\frac{3}{2}$. **19.** $-\frac{1}{2}\pm\frac{1}{2}\sqrt{-23}$.

20. 3, $\frac{1}{3}$, $-\frac{1}{4}\pm\frac{1}{4}\sqrt{-3}$, $-\frac{3}{2}\pm\frac{3}{2}\sqrt{-3}$. **21.** 2, 3, −1, −2.

22. 1, 1, $-\frac{3}{2}\pm\frac{1}{2}\sqrt{5}$.

Page 277

25. $\frac{n-m}{n+m}$. $\frac{n+m}{m-n}$. **26.** −4, 3, $-\frac{1}{2}\pm\frac{1}{2}\sqrt{-15}$. **27.** 5, $-\frac{5}{2}\pm\frac{5}{2}\sqrt{-3}$.

29. $a+b\pm\sqrt{a^2+ab+b^2}$. **30.** $\frac{d}{c}$, $-\frac{b}{a}$. **31.** 15. **32.** $\frac{2b}{a-c}$, $\frac{-b}{a+c}$.

33. −1, −2, −4, −8. **34.** 12. **35.** $-a$, $-b$.

36. $a+b$, 0, $\frac{a^2+b^2}{a+b}$. **37.** 5.10 p.m. **38.** $a\pm\frac{1}{a}$. **39.** $\frac{3c}{a+b}$, $\frac{-2c}{a+b}$.

Page 277 (*continued*)

40. 6076 nearly. **41.** $\frac{-v\pm\sqrt{v^2+64s}}{32}$. **42.** 1

43. $-\frac{1}{2}\pm\frac{1}{2}\sqrt{21}, -\frac{1}{2}\pm\frac{1}{2}\sqrt{13}$ **44.** 10 in. from a corner. **46.** 27.

Page 281

7. $(5, 2), (-\frac{14}{3}, \frac{47}{9})$. **8.** $(3, 2), (\frac{9}{13}, \frac{46}{13})$. **9.** $(6, 4), (-\frac{62}{11}, -\frac{5}{11}$

10. $(4, 1), (-\frac{7}{5}, -\frac{22}{5})$. **11.** $(2, 1), (-5, -\frac{11}{3}$

12. $(2{\cdot}525, {\cdot}175), (-2{\cdot}275, -1{\cdot}425)$. **19.** $4{\cdot}196, 4{\cdot}732$

20. $(-2, -1), (-\frac{7}{5}, \frac{1}{5})$.

Page 283

6. $(2, 1), \left(\frac{49\pm21\sqrt{5}}{2}, \frac{-7\mp3\sqrt{5}}{2}\right)$.

7. $(4, 3), (-1, -\frac{3}{2}), (3, 2), (-1, -\frac{3}{2})$.

8. $(1, 2), (-5, -10), \left(\frac{-3\pm\sqrt{89}}{4}, \frac{-9\pm3\sqrt{89}}{16}\right)$.

9. $(2, 4), (3, 3), (2, -3), (-3, -3)$.

Page 284

10. $(\pm4, \pm2), \left(\pm\frac{22}{\sqrt{13}}, \mp\frac{12}{\sqrt{13}}\right)$. **11.** $(\pm1, \pm2), \left(\pm\frac{\sqrt{13}}{13}, \mp\frac{10\sqrt{13}}{13}\right)$.

12. $(\pm3, \mp\frac{1}{2}), \left(\pm\frac{23}{\sqrt{65}}, \pm\frac{11}{2\sqrt{65}}\right)$. **13.** $(\pm4, \pm1), (\pm13\sqrt{\frac{5}{65}}, \pm5\sqrt{\frac{5}{6}}$

14. $(\pm1, \pm2), \left(\pm\frac{4}{\sqrt{3}}, \mp\frac{5}{\sqrt{3}}\right)$. **15.** $(0, 0), (1, 1)$, ([illegible]).

16. $(\pm6{\cdot}32, \pm3{\cdot}16)$. **17.** 35.

Page 286

16. $(\pm 2, \pm 1), (\pm 1, \pm 2)$. **17.** $(\pm 3, \pm 2), (\pm 2, \pm 3)$.

18. $(\pm 2, \pm 1), (\pm 1, \pm 2), (\pm\sqrt{-1}, \mp 2\sqrt{-1}), (\pm 2\sqrt{-1}, \mp\sqrt{-1})$

19. $(5, 2), (-\frac{1}{2}, -\frac{7}{2})$. **20.** $(6, 2), (-2, -6), (\frac{3}{2}\pm\frac{1}{2}\sqrt{57})(-\frac{3}{2}\pm\frac{1}{2}\sqrt{57})$.

21. $(5, 3), (3, 5), (6, 2), (2, 6)$. **39.** 7·32, ·68. **40.** $\frac{1}{4}b+\frac{1}{4}\sqrt{b^2-16a}$.

Page 291

13. $(2, 2), (2, 1), (1, 2), (2\pm\sqrt{2}, 2\mp\sqrt{2}), (\frac{3}{2}\pm\frac{1}{2}\sqrt{-7}, \frac{3}{2}\mp\frac{1}{2}\sqrt{-7})$.

19. 81. **21.** $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. **23.** $\left(a, \frac{b}{3}\right), \left(\frac{a}{3}, b\right)$. **24.** $\frac{4}{5}, \frac{7}{4}$.

27. $(\pm 10, \pm 5), (\pm 5\sqrt{2}, \pm 5\sqrt{2})$.

36. $(3, 2), (2, 3), (-2\pm\sqrt{-2}, -2\mp\sqrt{-2})$.

39. $(-1, 2), (2, -1), (-\frac{1}{2}\pm\frac{1}{2}\sqrt{13}, -\frac{1}{2}\pm\frac{1}{2}\sqrt{13})$.

40. $\frac{s+\sqrt{2d^2-s^2}}{2}, \frac{-s+\sqrt{2d^2-s^2}}{2}$. **41.** $(4, 1), (2, 2), (\frac{1}{3}, 12), (\frac{2}{3}, 6)$.

42. $(4, 2), (2, 4), (8, 1)$. **43.** $(5, 1), (1, 5)$.

Page 296

19. $\frac{bc}{a^2}$. **20.** x^{4a+4b}. **21.** 1. **22.** 1. **23.** a^n. **24.** 1.

25. $2^{3n}, 3^{12}$. **26.** 3. **27.** 2, 9 **28.** 2, 7, 3, 2.

Page 301

45. $\frac{1}{8}$ **46.** 8. **47.** 625. **48.** $11\frac{1}{9}$. **49.** 125. **50.** $\frac{1}{4}$.

51. $\frac{1}{16}$. **52.** 32. **53.** 4. **54.** $\frac{8}{7}$. **55.** $\frac{4}{3}$. **56.** $\frac{12}{35}$.

57. $\frac{8}{27}$. **58.** $\sqrt[3]{a^2b^3}$. **59.** 16, 8, 81, $\frac{1}{9}$, $\frac{1}{4}$.

Page 302

1. $x+x^{\frac{1}{2}}-6.$ **2.** $x^{\frac{3}{2}}-1$ **3.** x^3-1 **4.** $3x^2-8x^{\frac{3}{2}}+9x-10x^{\frac{1}{2}}.$

5. $a-1+4a^{-\frac{1}{2}}-4a^{-1}.$ **6.** $a^2-2a^{\frac{3}{2}}+3a-2a^{\frac{1}{2}}+1.$

7. $x^{\frac{5}{4}}+4x-11x^{\frac{3}{4}}-6x^{\frac{1}{2}}.$ **8.** $x^2+8x^{\frac{3}{2}}+24x+32x^{\frac{1}{2}}+16.$

9. $x^2+xy+y^2.$ **10.** $a^{\frac{3}{2}}-3a+3a^{\frac{1}{2}}-1.$ **20.** $x^2+x+1+x^{-1}+x^{-2}.$

21. $5a^{2m}+4a^m-2.$ **22.** $2x^2+6x+2.$ **23.** $1-2a.$

24. $x-2\sqrt{x}+3.$

Page 304

1. $x-4,\ a^2-b^2.$ **2.** $a+1+a^{-1}.$

3. $x^2-2x^{\frac{3}{2}}-x+2x^{\frac{1}{2}}+1,\ 4a^2-8a+4a^{-1}+a^{-2}.$

4. $a^{\frac{3}{2}}+3a+3a^{\frac{1}{2}}+1,\ 1-3x^{\frac{1}{2}}+3x-x^{\frac{3}{2}}.$ **5.** $x^4+x^2y^2+y^4.$

6. $x^{\frac{1}{3}}+y^{\frac{1}{3}}.$ **7.** $a^{\frac{1}{2}}+b^{\frac{1}{2}}+c^{\frac{1}{2}}.$ **8.** $(x+y)(x^{\frac{1}{2}}-y^{\frac{1}{2}})(x^{\frac{1}{2}}+y^{\frac{1}{2}}).$

9. $a^{\frac{1}{2}}-b^{\frac{1}{2}}.$ **10.** $\dfrac{x^{\frac{1}{2}}+3}{x^{\frac{1}{2}}-3},\ a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}.\ a-\sqrt{ab}+b.$

11. $x^{\frac{1}{2}}-2,\ x-x^{\frac{1}{2}}+1.$ **12.** $2x^{-2}+3x^{-1},\ x+2-x^{-1}.$

Page 305

3. $5,\ \frac{7}{12},\ 49.$ **4.** $4,\ \frac{1}{27},\ 25,\ 4,\ \frac{1}{64},\ 8,\ \frac{1}{16}.$ **6.** $5\frac{1}{16},\ 2.$

7. 3·162, 1·778, 1·333, 5·62. **8.** 4. **9.** 1·732.

10. $9\frac{3}{8},\ -1\frac{11}{16}.$ **11.** 100. **12.** 4, 2. **13.** $\frac{1}{3},\ 1\frac{1}{3}.$

14. $\frac{2}{3},\ \frac{2}{7}.$ **15.** $\dfrac{\sqrt{a}+5}{\sqrt{a}+4},\ 3x^{\frac{1}{4}}+2,\ \dfrac{a}{a^{\frac{1}{2}}b-b^2}.$ **16.** $x^2y^{\frac{1}{2}}+3x^{\frac{1}{2}}y^2.$

Page 305 (*continued*)

17. $x^{\frac{2}{3}}y^{-1}+1+x^{-\frac{2}{3}}y.$ **18.** $x^{\frac{2}{3}}-y^{-\frac{2}{3}},\ a^{\frac{1}{3}}-1+a^{-\frac{1}{3}}$

19. $a^{3m}+a^{2m}b^{m}+a^{m}b^{2m}+b^{3m}.$ **20.** $y+2y^{\frac{1}{2}}+1.$ **21.** $x-2-x^{-1}.$

22. $\frac{1}{2}a^{x}-\frac{2}{3}b^{y}.$ **24.** ·0016, 1·44, 3·375, 8. **25.** $\frac{1}{4}\left(a-\frac{1}{a}\right), \frac{1}{2}\left(a+\frac{1}{a}\right).$

26. 1. **27.** $e^{x}+e^{-x},\ x^{\frac{3}{2}}-2xy^{\frac{1}{2}}+3x^{\frac{1}{2}}y-y^{\frac{3}{2}}.$ **28.** $abc.$

30. 4, 32. **31.** 2750. **32.** $\frac{1}{4}.$ **33.** 2, 3.

34. $x^{\frac{2}{3}}+2x^{\frac{1}{3}}+1+2x^{-\frac{1}{3}}+x^{-\frac{2}{3}}.$ **36.** $2x+x^{\frac{1}{2}}-x^{-\frac{1}{4}}.$

Page 310

7. $\sqrt[6]{4}, \sqrt[6]{27};\ \sqrt[12]{16}, \sqrt[12]{27};\ \sqrt[12]{64}, \sqrt[12]{81}, \sqrt[12]{125}.$

8. $3\sqrt{2}, 5\sqrt{6}, \sqrt{5}$, 1·26, $\sqrt[4]{5}.$ **9.** $12\sqrt{2}.$ **10.** $12\sqrt{5}.$

11. $33\sqrt{2}.$ **12.** $3\sqrt[3]{2}.$ **13.** $7\sqrt[3]{12}.$ **14.** $10\sqrt[4]{2}.$ **15.** $9\sqrt{3}.$

16. 0. **17.** $\sqrt{3}, \sqrt{5}, x\sqrt{xy}, y\sqrt[3]{4x}, \sqrt{2}.$

18. 2·52, 3·78, 12·6, ·63, ·126, 1·26.

Page 311

25. $3\sqrt{10}.$ **26.** $\frac{1}{5}\sqrt{5}.$ **27.** $\frac{2}{5}\sqrt{5}.$ **28.** $\sqrt{2}.$

29. $\frac{1}{5}(2\sqrt{2}-\sqrt{3}).$ **30.** $\sqrt{a^{2}+b^{2}}+b.$ **31.** $\sqrt{a+b}-\sqrt{c}.$

32. $\frac{1}{y}(x-\sqrt{x^{2}-y^{2}}).$ **33.** 2·517, 1·354. **34.** 194.

35. $27(\sqrt{3}-\sqrt{2}).$ **36.** $\frac{17}{22}(7-\sqrt{5}), 2\sqrt{5}.$ **37.** $\frac{1}{11}(18-34\sqrt{5})$

39. 10 ft. 5 in.

Page 314

16. No root. **17.** 4. **18.** $\frac{11}{36}$. **19.** 100. **20.** 9.
21. a. **22.** 25. **23.** 64. **24.** No root. **25.** $\frac{(a-b)^2}{2a-b}$.
26. 3. **27.** 10. **28.** 10. **29.** $\frac{2}{5}$. **30.** $\frac{2ac}{c^2+1}$

Page 317

18. (4, 9), (9, 4). **19.** (4, 16), (16, 4). **20.** (17, 8).
21. (9, 1), (1, 9). **22.** $(2, \frac{1}{2})$, $(\frac{1}{2}, 2)$. **23.** 2, 1. **24.** 7, -6.
25. (2, 8), (8, 2).

Page 320

10. 2·823. **11.** ·196.
12. $2^{\frac{1}{4}}(2+\sqrt{3})$, $5^{\frac{1}{4}}(\sqrt{5}+1)$, $3^{\frac{1}{4}}(2-\sqrt{3})$, $2^{\frac{1}{4}}(5\sqrt{2}+3)$. **14.** $1+\sqrt{3}$.
15. $\frac{\sqrt{3}-1}{\sqrt{2}}$, $\frac{\sqrt{2}+1}{\sqrt{2}}$, $\frac{\sqrt{7}+2}{\sqrt{3}}$. **17.** 2·309. **18.** $2+3\sqrt{2}$.

Page 322

10. $25\sqrt{-3}$. **11.** 68. **12.** -25. **13.** $1+\sqrt{-1}$.
14. $\frac{-1-\sqrt{-3}}{2}$. **15.** $2a^3-2b^3$.

Page 324

1. $2+2\sqrt{2}-2\sqrt{3}$. **2.** $2+\frac{5}{6}\sqrt{6}$. **4.** 1·98, 3·15, 1·39, 3·55.
5. 9. **6.** $\frac{37}{36}$. **7.** $7\frac{1}{5}$. **8.** 7, -1. **9.** 20. **10.** 13.
11. $6+2\sqrt{15}$, $x^2+2xy+y^2-4x-4y$. **12.** $\frac{7}{4}\sqrt{3}$.
13. $4a+2\sqrt{4a^2-b}$ **14.** $\frac{1}{2q}(p^2-2pq+q^2)$. **15.** 12.

Page 324 (*continued*)

16. x^4-7x^2+2x+2. **17.** 0. **18.** $2a$, $4a^2-2$, $8a^3-6a$.

19. 4, 7. **20.** $\sqrt{3}$, $\dfrac{2m}{\sqrt{m^2-n^2}}$. **21.** 5.

22. $\dfrac{\sqrt{13}+1}{\sqrt{2}}$, $\sqrt{a+1}+\sqrt{a-1}$. **23.** 4, -7. **24.** 40.

25. 2·62, ·38. **26.** $\dfrac{2}{b}\sqrt{a^2-b^2}$. **29.** 1. **30.** $10\sqrt{2}$.

31. $16+9\sqrt{3}$.

Page 330

17. $x^2-2mx+m^2-n^2=0$. **18.** $x^2-4ax+4a^2-b^2=0$.

19. $x^2-6x+6=0$. **20.** $16x^2+8x-63=0$.

21. $x^3-28x-48=0$. **22.** $24x^3-26x^2+9x-1=0$.

24. $4x^2-28x+45=0$. **25.** $x^2-7x+12=0$.

26. $x^2(a^2-b^2)-2x(a^2+b^2)+a^2-b^2=0$, $4x^2-16x+9=0$.

27. 9, 7 ; $a+b$, 0 ; $2p$, pq ; $2c-2a-2b$, $a^2+b^2-c^2$.

28. 0, 5, -1, -4. **29.** 4, 8. **30.** ±16.

Page 334

1. $1\frac{2}{3}$, $6\frac{1}{3}$, 22. **2.** 47, $-1\frac{4}{9}$. **3.** $-2\frac{5}{6}$, $1\frac{1}{6}$, $6\frac{4}{9}$.

4. $-5\frac{2}{3}$, $1-3a$. **5.** $x^2-18x+80=0$.

6. $x^2-5x+4=0$, $x^2+5x+5=0$. **7.** $6x^2-x-2=0$, $qx^2-px+1=0$.

8. $5x^2-2x+3=0$, $15x^2+26x+15=0$, $9x^2+26x+25=0$.

9. a^2-2b, $3ab-a^3$. **10.** $x^2-x(p^2+2q)+q^2=0$.

11. $ax^2-x(2ah-b)+ah^2-bh+c=0$. **12.** $x^2-4x-4=0$.

14. $x^2+6x+8=0$. **16.** $\dfrac{1}{a^4}(b^2-ac)(b^2-3ac)$.

Page 338

6. Rational. **7.** Real and irrational. **8.** Imaginary.
9. Real and equal. **10.** Rational **11.** Real. **13.** 4.
14. ± 5. **16.** $-\frac{1}{2}$. **19.** $\dfrac{a}{m}$. **20.** $2, -\frac{3}{7}$.

Page 340

8. $(x+2+\sqrt{7})(x+2-\sqrt{7})$. **9.** 16. **10.** $\pm 6a$.
11. $(x-3+2\sqrt{5})(x-3-2\sqrt{5})$.
12. $(3x-4y)(3x+4y)(4x-3\ )(4x+3y)$; $\pm\frac{4}{3}, \pm\frac{3}{4}$.
13. 174; $(8x+7)(15x-4)$. **14.** $b^2=4ac$.

Page 341

4. $6, \frac{6}{25}, \frac{2}{3}, 4, -14$. **5.** $\frac{5}{2}, \frac{1}{2}$. **6.** $16x^2-40x+21=0$. **7.** $12\frac{7}{8}$.
8. $6x^2-19x+15=0$. **9.** -25. **10.** $x^2\pm 12x+35=0$.
11. $(72x+1)(73x-1)$, $(13x+11)(17x-15)$. **12.** $x^2-4x+3=0$.
13. $2a+2b-2c$. **14.** $97x^2-53x-17=0$.
15. $(x+3+\sqrt{2})(x+3-\sqrt{2})$. **16.** $2x^2-17x=0$.
17. $ax^2+3bx+9c=0$. **18.** $acx^2-x(b^2-2ac)+ac=0$.
19. $1, \dfrac{a+b-c}{a+b+c}$. **23.** ± 12. **26.** ± 4. **27.** $3mn-m^3$. **28.** a.
29. $\dfrac{2ab}{a+b-c}, \dfrac{abc}{a+b-c}$. **30.** $\frac{1}{2}$. **31.** $c+b-a$. **32.** $a=8$ or 0.
33. 8, 1. **35.** 6, 2 **36.** $(x+b+\sqrt{b^2-c})(x+b-\sqrt{b^2-c})$.

Page 345

16. $2a-b+3$. **17.** $\frac{p-7q+3r}{p-q+r}$. **18.** $-7, 1, 19$.

19. $(x-y+3)(x+2y-4)$. **20.** $2a-3, 3a-4$

21. $(2a-b+c)(3a-b-c)(3a+2b-2c$.

Page 348

13. $(2x-y-5z)(4x^2+y^2+25z^2+2xy+10xz-5yz)$.

14. $(a+b+c+1)(a^2+b^2+c^2+2ab-ac-bc-a-b-c+1)$.

15. $a^3-b^3-c^3-3abc$. **16.** $8x^3-y^3+27z^3+18xyz$.

17. $1-a^3-b^3-3ab$. **18.** $8a^3-27b^3-64-72ab$.

19. $1+a^2+b^2+a-b+ab$. **20.** $9m^2+n^2+1+3mn+3m-n$.

21. $a+5b-1$. **22.** $3a+b$. **23.** x^2-3x+9. **24.** $2a-5$.

25. $a+b+c+d-1$. **28.** $27x^3-8y^3+z^3+18xyz$. **35.** 0.

36. $\frac{a^2+b^2+c^2+ab+ac-bc}{2a-3b+3c}$, $\frac{x+2y+z}{2}$. **37.** $(x+y+z)(a+b)$.

40. a, b.

Page 351

9. $(x-y)(y-z)(x-z)$. **10.** $(x-y)(y-z)(x-z)$.

11. $(x-y)(y-z)(z-x)$. **12.** $(a-b)(b-c)(c-a)(a+b+c)$.

13. $(a-b)(b-c)(c-a)(a+b)(b+c)(c+a)$. **14.** $a+b+c$.

15. $\frac{c}{a}, \frac{d}{b}$. **16.** $\frac{a+b}{a-b}, \frac{b-a}{b+a}$. **17.** a, b. **18.** $\frac{a+b}{a}, \frac{a-b}{b}$.

19. $\frac{b}{a}, \frac{b+a}{b-a}$. **20.** $ax-a-b$.

Page 353

10. $x^4-2x^3+4x^2-8x+16.$ **11.** $x^3-3x^2+9x-27.$

12. $x^8+x^6a+x^4a^2+x^2a^3+a^4.$ **13.** $(a+b)^3-(a+b)^2+a+b-1.$

14. $x-b,\ a+b,\ x-4,\ m+\frac{1}{m},\ x+y-1.$ **15.** $a^4-1.$ **16.** $m^6+1.$

17. $a^6-b^6.$ **20.** $a^2+ap+q=0.$ **22.** 1, 9.

23. $x^{\frac{2}{3}}+x^{\frac{1}{3}}a^{\frac{1}{3}}+a^{\frac{2}{3}},\ x^{\frac{2}{3}}-x^{\frac{1}{3}}a^{\frac{1}{3}}+a^{\frac{2}{3}},\ x^{\frac{4}{5}}+x^{\frac{3}{5}}a^{\frac{1}{5}}+x^{\frac{2}{5}}a^{\frac{2}{5}}+x^{\frac{1}{5}}a^{\frac{3}{5}}+a^{\frac{4}{5}},$
$x^{\frac{4}{5}}-x^{\frac{3}{5}}a^{\frac{1}{5}}+x^{\frac{2}{5}}a^{\frac{2}{5}}-x^{\frac{1}{5}}a^{\frac{3}{5}}+a^{\frac{4}{5}}.$

Page 357

13. $3(a^2+b^2+c^2)-2(ab+bc+ca).$ **14.** $2(a^2+b^2+c^2).$ **15.** 0.

16. $-3(a^2b-ab^2+b^2c-bc^2+c^2a-ca^2).$ **17.** $(x-y)(y-z)(x-z).$

18. $(x-y)(y-z)(x-z).$ **19.** $(a+b)(b+c)(c+a).$ **20.** $24abc.$

21. $3(x-y)(y-z)(z-x).$ **22.** $(a+b)(b+c)(c+a).$

23. $-(a-b)(b-c)(c-a)(a+b+c).$

24. $(a-b)(b-c)(c-a)(a+b)(b+c)(c+a).$ **25.** 1. **26** 1.

27. 1. **28.** $-\frac{1}{abc}.$ **29.** 0. **30.** $a+b+c.$ **31.** $-(x+y+z).$

32. 3. **33.** $6abc$ **35.** $3(x^2+y^2+z^2)+2(xy+yz+zx).$

Page 362

24. 18. **26.** $a^2-2b^2.\ a^3-3ab^2.$ **27.** $a^2=c^2+2b^2.$

28. $a^3+2c^3=3ab^2$

Page 362

8. $\frac{1}{8}x^3+\frac{1}{27}y^3-\frac{1}{64}z^3+\frac{1}{8}xyz$. **9.** $x=(a-1)(1-b)$. **11.** $\frac{1}{abc}$.
12. $\frac{1}{3}(a+b+c)$. **15.** $-(ab+bc+ca)$. **16.** $\frac{x}{abc}$. **18.** $-\frac{1}{3}(a+b+c)$.
22. $(x-1)(x^3-2x^2+2x+2)$. **23.** $-a-b-c$. **28.** $(x+1)(a+b+c)$.
29. $3(x^2+y^2+z^2)+\frac{1}{2}(xy+yz+zx)$. **30.** $a^2=2b+c$. **31.** 9, 17.
32. 38, 70.

Page 367

1. 23, 58. **2.** −9, −44. **3.** $4\frac{1}{2}$, $11\frac{1}{2}$. **4.** $-2\frac{1}{2}$, −13. **5.** $9a$, $23a$.
6. $-{\cdot}4$, $-3{\cdot}2$. **7.** $a-3b$, $a-10b$. **8.** $6a-5b$, $13a-12b$.
9. $5x+9$, $12x+23$. **10.** $-3(m+n)$, $-10(m+n)$. **11.** 199, $93\frac{1}{2}$.
12. $10\sqrt{2}$, $5\sqrt{5}$. **13.** n. **14.** $2n-1$. **15.** $3n+4$. **16.** $5-3n$.
17. $5x-2nx$. **18.** 0. **19.** $a^2+4ab+b^2-2abn$. **20.** 1. **21.** 20th.
22. 100th. **23.** 15th. **24.** 23rd. **25.** 4. **26.** 260. **27.** 40.
28. −110. **29.** 26th. **30.** 91. **31.** 57. **32.** 3, 4. **33.** 6.
34. 123. **35.** 64. **37.** 10. **38.** 3, 5, 7. **39.** 46. **41.** 15, 20.

Page 370

1. 1600. **2.** 400. **3.** −400. **4.** $169a$. **5.** $2nx$. **6.** $2an$.
7. 590. **8.** 372. **9.** −2592. **10.** $265\frac{1}{2}$. **11.** $-7\frac{1}{2}$. **12.** 99.
13. $22\frac{2}{3}$. **14.** −84. **15.** $144a$. **16.** 595. **17.** $\frac{3}{2}n^2+\frac{1}{2}n$.
18. $\frac{1}{2}n^2+\frac{1}{2}n$. **19.** $3n^2+2n$. **20.** $4n-n^2$. **21.** $5\frac{1}{4}n-1\frac{3}{4}n^2$.
22. $2an^2-an$. **23.** $an+n^2$. **24.** $\frac{1}{2}n-\frac{1}{2}$. **25.** n^2. **26.** 2120.
27. 8 or 9. **28.** 15 or 22. **29.** 17. **30.** 0. **31.** n^2+n.
32. 1, 3, 5, ... **33.** 17. **34.** 1560 yd. **36.** 250500. **37.** 3340.
38. 2318·4 ft. **39.** 7. **40.** 32. **41.** 19th. **42.** −1, 3, 7, ...

Page 372

2. $d=6$; $\frac{2}{3}$; $1\frac{1}{6}$; $-a$. **3.** $d=3$, $S=670$. **4.** 313.
5. $x+\frac{r(y-x)}{m+1}$. **6.** 1, -2. **8.** $\frac{1}{3}(2a+b)$, $\frac{1}{3}(a+2b)$.

Page 375

1. 15, 12. **2.** $\frac{1}{6}$, $\frac{1}{9}$. **3.** [illegible]. **4.** 12, 24. **5.** $-1\frac{2}{3}$, $-1\frac{1}{5}$.
6. $\frac{1}{7}(3-\sqrt{2})$, $\frac{1}{14}(4-\sqrt{2})$. **7.** 1. **8.** $\frac{4}{9-n}$. **9.** $\frac{ab}{b+(n-1)(a-b)}$.
10. 147. **11.** 840, 420. **12.** 16th. **13.** 20, -24, $\frac{1}{8}$, $\frac{a^2-b^2}{a^2+b^2}$.
14. $1\frac{1}{7}$, $1\frac{1}{3}$, $1\frac{3}{5}$; $3\frac{1}{2}$, $1\frac{2}{5}$, $\frac{7}{8}$, $\frac{7}{11}$; $1\frac{1}{2}$, 1, $\frac{3}{4}$, $\frac{3}{5}$, ... $\frac{3}{11}$. **16.** 10, 15.
17. 20, 12 or 3, -5. **19.** $\frac{3ab}{a+2b}$, $\frac{3ab}{2a+b}$. **20.** 3. **21.** $\frac{ab+bc-2ac}{a+c-2b}$.

Page 378

1. 512. **2.** $\frac{1}{81}$. **3.** $1\frac{5}{27}$. **4.** $-\frac{5}{16}$. **5.** x^{10}. **6.** $-\frac{1}{64}$. **7.** $b^{10}a^{-8}$.
8. $a^{21}b^{28}$. **9.** x^{-4}, x^{3-n}. **10.** -2^{1-2n}, 2^{-2n}. **11.** $(a+b)^{n-1}(a-b)^{3-n}$.
12. $\frac{1}{3}$. **13.** $a=10$, $r=\pm 2$. **14.** 8th. **15.** $-\frac{1}{128}$. **16.** 2.
17. 6, 27, 2, a^2-b^2. **18.** 15, 45. **19.** $r=2$; $\frac{2}{\sqrt{3}}$; y. **20.** 192.
21. 4, 16. **22.** 5, 125. **23.** $\frac{1}{2}$, 2, 8,.... **29.** 3, 6, 12.

Page 382

1. 511. **2.** $1\frac{31}{32}$. **3.** $10\frac{2}{3}$. **4.** $10\frac{11}{16}$. **5.** $40(\sqrt{3}+1)$. **6.** ·111111.
7. $\frac{1}{3}(1-2^{2n})$. **8.** $\frac{1}{3}(2+2^{-2n})$. **9.** $\frac{a}{2a-1}(2^na^n-1)$.
10. 20·0235. **11.** 8190. **12.** $46\frac{2}{3}$, $12\frac{2}{3}$. **14.** 32. **15.** $1\frac{1}{3}$.
16. $\frac{2}{3}$. **17.** 15. **18.** $4\frac{1}{2}+\frac{1}{2}\sqrt{3}$. **19.** $\frac{5}{6}$. **20.** $\frac{2}{3}a$.

Page 382 (*continued*)

21. $\frac{1}{3}(25+5\sqrt{15})$. **22.** 21. **23.** $\frac{a}{1+r}$. **24.** $\frac{1}{3}$. **25.** $\frac{1}{6}$.
26. $\frac{23}{110}$. **27.** $\frac{4}{9}$, $\frac{13}{99}$, $\frac{13}{900}$. **28.** 41. **30.** $\frac{1}{1024}$.
32. $a=24$, $r=-\frac{1}{3}$. **33.** 18, 12. **34.** $x<2\frac{1}{3}$. **35.** 224 ft.

Page 383

1. $3\frac{1}{2}$, 4; 5, 10; 3[illegible], 4[illegible]. **2.** 10. **4.** 5. **5.** 7, 9, 11.
7. 6 or 13. **8.** $\frac{1}{1458}$. **9.** 247500. **11.** ±8. **12.** $\frac{3}{2}n^2-\frac{1}{2}n$.
13. 11, 17, 23. **17.** 12 days. **19.** 2. **23.** $\frac{1}{3}, \frac{1}{2}, 1, \ldots\ldots$ **24.** 9.
25. $\frac{1}{2}$, $\frac{1}{4}$ or $\frac{7}{8}$, $-\frac{7}{8}$. **26.** 45. **28.** $a\frac{a^n-1}{a-1}+\frac{nb}{2}(n+1)$.
29. $\frac{(1+a)(a^nb^n-1)}{ab-1}$.

Page 386

1. 20. **2.** 4. **3.** 72. **4.** 200. **5.** 12. **6.** 30. **7.** 650, 15600.
8. 380. **9.** 720.

Page 389

1. 210, 840, 720. **3.** 720. **4.** 5040. **5.** 1680. **6.** 7.
7. 4. **8.** 720. **9.** 5040, 3600. **10.** 60, 325. **11.** 86400.
12. 96. **13.** $\underline{|6}-\underline{|4}=696$.

Page 391

1. 729. **2.** 2520, 420, 30240, 1663200. **3.** 420. **4.** 12.
5. 10. **6.** 120, 60, 21, 6. **7.** 3024. **8.** 1260. **9.** 256.

Page 394

1. 10, 28, 4845. **2.** 20, 455, 1287, 120. **3.** 220. **4.** 6. **5.** 99. **6.** 8. **7.** 136, 1140. **8.** 28. **9.** 20, $\frac{1}{2}n(n-3)$. **10.** $\frac{1}{6}n(n-1)(n-2)$. **11.** 7056. **12.** 5400. **13.** 1092000. **14.** 2211. **15.** 31. **17.** 2002, 3003.

Page 396

1. 6840, 8000. **2.** $(n-2)\underline{|n-1}$. **3.** 15. **4.** 40320, 5040. **5.** 3024, 6561. **6.** 600. **7.** 1554. **8.** 60. **9.** 4320. **10.** $\frac{1}{2}n(n-1)$. **11.** n, n or $n+1$. **12.** 11. **13.** 1035. **14.** 252. **15.** 1260. **16.** 462. **17.** $\underline{|52} \div (\underline{|13})^4$. **18.** 371. **19.** 792. **21.** 243 **22.** 67200. **23.** 3999960.

2½/05